Bergen County Academies Math Competition (Grades 7-8)
October 26th, 2003

Note. The first fifteen problems for the two exams were identical, but in different orders.

1. A number plus 4 is 2003. What is the number?
   The number is $2003 - 4 = 1999$.

2. Jane has 4 pears, 5 bananas, 3 lemons, 1 orange, and 6 apples. If she uses one of each fruit to make a fruit smoothie, what is the total number of fruits that she has left?
   Jane has $(4-1)+(5-1)+(3-1)+(1-1)+(6-1) = 3+4+2+0+5 = 14$ fruits left.

3. Compute $\frac{55}{21} \times \frac{28}{5} \times \frac{3}{2}$.
   Simplifying the fractions, we have $\frac{55}{21} \times \frac{28}{5} \times \frac{3}{2} = \frac{55\times28\times3}{21\times5\times2} = \frac{5\times11\times2\times2\times7\times3}{3\times7\times5\times2} = 11 \times 2 = 22$.

4. Philip has 3 triangles and 6 pentagons. Let S be the total number of sides of the shapes he has. Let N be the number of shapes he has. What is $S + N$?
   Since a triangle has 3 sides and a pentagon has 5 sides, $S = (3\times3)+(6\times5) = 39$. Also, since Philip has $3+6=9$ total shapes, $N = 9$. Therefore $S + N = 39 + 9 = 48$.

5. On an exam with 80 problems, Roger solved 68 of them. What percentage of the problems did he solve?
   Roger solved $\frac{68}{80} \times 100\% = 85\%$ of the problems.

6. Yao Ming is 7 feet 5 inches tall. His basketball hoop is 10 feet from the ground. Given that there are 12 inches in a foot, how many inches must Yao jump to touch the hoop with his head?
   Yao is $(7\times12)+5 = 89$ inches tall, but the hoop is $10\times12 = 120$ inches from the ground. Therefore, he must jump $120-89 = 31$ inches.

7. In triangle ABC, $\overline{BC} = 4$ and $\overline{CA} = 6$. If the perimeter of the triangle is 4 times the length of side $\overline{BC}$, what is the length of $\overline{AB}$?
   The perimeter is obviously $4\times4 = 16$, so $\overline{AB} = 16-4-6 = 6$.

8. Let A be the sum of seven 7's. Let B be the sum of seven A's. What is B?
   From the given, $A = 7\times7 = 49$ and $B = 7\times A = 7\times49 = 343$.

9. Compute $664.02 \div 9.3$.
   Dividing directly, we have $664.02 \div 9.3 = 71.4$.

10. If a certain number is doubled and the result is increased by 11, the final number is 23. What is the original number?
    Let the number be x. From the given, we obtain the equation $2x+11 = 23$, and solving for x, we see that $x = 6$.

11. Compute the product of the integers from -5 to 5, inclusive.
    Note that 0 is included in the product, so the answer is 0.

12. A problem author for a math competition was looking through a tentative exam when he realized that he could not use one of his proposed problems. Frustrated, he decided to take a nap instead, and slept from 10:47AM to 7:32PM. For how many minutes did he sleep?
    We first note that 7:32PM is equivalent to 19:32AM, which is in turn equivalent to 18:92AM. Now, we may subtract the hours and the minutes directly: 18:92 − 10:47 = 8:45. In other words, 8 hours and 45 minutes elapsed, which is equivalent to $(8\times60)+45 = 525$ minutes.

13. In rectangle $ABCD$, $\overline{AB} = 7$ and $\overline{AC} = 25$. What is its area?
    Using the Pythagorean Theorem in right triangle ABC, we see that $7^2 + BC^2 = 25^2$, and solving for BC yields $BC = 24$. Therefore, the area of the rectangle is $(AB)(BC) = (7)(24) = 168$.

14. Evaluate $\frac{100-99+98-97+\cdots+4-3+2-1}{1-2+3-4+\cdots+97-98+99-100}$.
First Solution. We can immediately obtain the answer by noting that the denominator is equivalent to the numerator, only with each sign changed. This implies that the denominator is equal to the negative of the numerator, so the answer is -1.
Second Solution. We compute the numerator first. Group the numbers into 50 pairs: $(100-99)+(98-97)+(96-95)+\cdots+(4-3)+(2-1)$, and note that the quantity inside each pair of parentheses is equal to 1. Therefore, the numerator is equal to 50. Similarly, group the denominator into 50 pairs: $(1-2)+(3-4)+(5-6)+\cdots+(97-98)+(99-100)$. Since the quantity inside each pair of parentheses is equal to -1, the denominator is equal to -50. Therefore, the answer is $\frac{50}{-50}=-1$.

15. What is the area of a square in square feet, if each of its diagonals is 4 feet long?
First Solution. Label the square $ABCD$, and let its side have length s. Applying the Pythagorean Theorem in right triangle ABC, we have $s^2+s^2=4^2$, or $s^2=8$. Since the area is equal to s^2, the answer is 8.
Second Solution. Regard the square as a rhombus with diagonals whose lengths are both 4. The area of a rhombus is equal to half the product of its diagonals, so the area of the square is simply $\frac{1}{2}(4)(4)=8$.

16. A lazy student used the approximation $\pi=\frac{22}{7}$ to calculate the circumference of a given circle. If his answer was 6, what was the radius of the circle?
The circumference c of a circle with radius r can be obtained using the formula $c=2\pi r$. Substituting the given values yields the equation $6=2\cdot\frac{22}{7}\cdot r$, and solving for r, we have $r=\frac{21}{22}$.

17. Find the largest divisor of 2800 that is a perfect square.
First Solution. Since 2800 is obviously divisible by 100, a perfect square, we write $2800=100\times 28$. It now suffices to find the largest square factor of 28, which is 4. Thus, the answer is $100\times 4=400$.
Second Solution. We first write 2800 as the product of primes. (In other words, we *prime factorize* it.) Now note that in the factorization $2800=2^4\times 5^2\times 7$, the prime 2 occurs four times and the prime 5 occurs two times. In the *prime factorization* of any perfect square, each prime must occur an even number of times. Therefore, the largest square factor that divides 2800 must be $2^4\times 5^2=400$.

18. How many multiples of 17 are there between 23 and 227?
First Solution. Simply count by 17's. The multiples of 17 begin 17, 34, 51, 68, 85, 102, 119, 136, 153, 170, 187, 204, 221, 238, and etc. Therefore, there are 12 multiples of 17 between 23 and 227.
Second Solution. Note that $23\div 17=1.35...$, so $23=17\times 1.35....$ Also, $227\div 17=13.35...$, so $227=17\times 13.35....$ The multiples of 17 between those two numbers are 17×2, 17×3, 17×4, ..., 17×12, and 17×13, so there are 12 of them.
Note. In general, the number of positive multiples of n, greater than a and less than or equal to b, is $\lfloor\frac{b}{n}\rfloor-\lfloor\frac{a}{n}\rfloor$, where $\lfloor x\rfloor$ denotes the greatest integer less than or equal to x.

19. Two angles are supplementary, and one angle is 9 times as large as the other. What is the number of degrees in the measure of the larger angle?
Let the measure of the larger angle be x, so the smaller angle is $\frac{x}{9}$. Since the two angles are supplementary, $x+\frac{x}{9}=180$, or $\frac{10}{9}x=180$. Therefore, $x=180\cdot\frac{9}{10}=162$, so the answer is $162\,^\circ$.

20. How many positive whole numbers less than 100 are divisible by 3, but not by 2?
The multiples of 3 less than 100 are 3, 6, 9, 12, …, 93, 96, and 99. Since $99=3\times 33$, there are 33 of them. However, we want only the multiples of 3 that are not even. In other words, we need to take out all the multiples of 6 from our count. The multiples of 6 less than 100 are 6, 12, 18, 24, ..., 84, 90, and 96. Since $96=6\times 16$, there are 16 of them. Therefore, the answer is $33-16=17$.

21. The surface area and the volume of a cube are numerically equal. Find the cube's volume.
Let s be the length of an edge of the cube. Then, the surface area of the cube is $6s^2$, and the volume of the cube is s^3. Setting these two quantities equal, we have the equation $6s^2=s^3$, and dividing both sides by s^2 yields $6=s$. Therefore, the cube's volume is $s^3=6^3=216$.

22. Given that $|3-a|=2$, compute the sum of all possible values of a.
For the absolute value of a given number to be 2, that number must be either 2 or -2. Therefore,

we simply need to solve the equations $3 - a = 2$ and $3 - a = -2$, and obtain $a = 1$ and $a = 5$ as the possible values of a satisfying the given equation. Therefore, the answer is $1 + 5 = 6$.

23. Let $ABCD$ be a square with side length 8. A second square $A_1B_1C_1D_1$ is formed by joining the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$. A third square $A_2B_2C_2D_2$ is formed in the same way from $A_1B_1C_1D_1$, and a fourth square $A_3B_3C_3D_3$ from $A_2B_2C_2D_2$. Find the sum of the areas of these four squares.
Note that each square we produce has half the area of its predecessor. Since the area of $ABCD$ is $8 \times 8 = 64$, the area of $A_1B_1C_1D_1$ is 32, the area of $A_2B_2C_2D_2$ is 16, and the area of $A_3B_3C_3D_3$ is 8. Therefore, the answer is $64 + 32 + 16 + 8 = 120$.

24. If $a + b = 13$, $b + c = 14$, $c + a = 15$, find the value of c.
First Solution. Add the three equations to obtain $2a + 2b + 2c = 42$, or $a + b + c = 21$. Now subtract the first given equation from this to obtain $c = 8$.
Second Solution. From the second equation, we have $b = 14 - c$, and from the third equation, we have $a = 15 - c$. Substituting these into the first equation, we have $(15 - c) + (14 - c) = 13$, and solving for c yields $c = 8$.
Note. The trick used in the first solution is a very famous one. Exploiting symmetry in general is a useful tactic.

25. Two positive whole numbers differ by 3. The sum of their squares is 117. Find the larger of the two numbers.
First Solution. Let the larger number be x. Then, the given implies that $x^2 + (x - 3)^2 = 117$. Simplifying, we have $2x^2 - 6x + 9 = 117$, or $x^2 - 3x - 54 = 0$. Factoring this expression yields $(x + 6)(x - 9) = 0$, and since x is positive, we must have $x = 9$.
Second Solution. We can also use trial and error. Try 5 and 8, and note that $5^2 + 8^2 = 25 + 64 = 89$, which is smaller than 117. Then, try 6 and 9, and note that $6^2 + 9^2 = 36 + 81 = 117$, which works. Therefore, the answer is 9.
Note. For this particular problem, trial and error is a faster method than solving it algebraically because it was given that the two numbers are natural, and the sum of their squares was relatively small.

26. Given that $5^3 + 5^3 + 5^3 + 5^3 + 5^3 = 5^J$ and $3^2 + 3^2 + 3^2 = 3^N$, what is the value of J^N?
Clearly, $5^3 + 5^3 + 5^3 + 5^3 + 5^3 = 5 \times (5^3) = 5^4$, so $J = 4$. Similarly, $3^2 + 3^2 + 3^2 = 3 \times (3^2) = 3^3$, so $N = 3$. Therefore, $J^N = 4^3 = 64$.

27. A pair of positive integers a and b is such that their greatest common divisor is 5 and their least common multiple is 55. Find the smallest possible value of $a + b$.
Clearly, a and b are both multiples of 5, and at least one of them must be a multiple of 11 as well. Then, the smallest possible value of $a + b$ obviously occurs when one is equal to 5 and the other equal to 55, and thus, the answer is $5 + 55 = 60$.

28. How many of the positive divisors of 120 are divisible by 4?
The number of divisors of 120 divisible by 4 is equal to the number of divisors of 30. (This is because if $4n$ divides 120, n clearly divides 30, and vice versa.) The divisors of 30 are 1, 2, 3, 5, 6, 10, 15, and 30, so the answer is 8.

29. How many three-digit numbers are perfect squares?
First, note that $31^2 = 961$ and $32^2 = 1,024$. Therefore, the three-digit perfect squares are 10^2, 11^2, 12^2, ..., 30^2, and 31^2, so the answer is 22.

30. Calculate $1 + 3 + 5 + \cdots + 195 + 197 + 199$.
We can write the sum in pairs: $(1 + 199) + (3 + 197) + (5 + 195) + \cdots + (97 + 103) + (99 + 101)$. Since there are 50 pairs, and each pair adds up to 200, the answer is $50 \times 200 = 10,000$.
Note. Using this logic, one can prove that the sum of the first n odd positive integers is n^2.

31. The ages of Mr. and Mrs. Fibonacci are both two-digit numbers. If Mr. Fibonacci's age can be formed by reversing the digits of Mrs. Fibonacci's age, find the smallest possible positive difference between their ages.
Let the tens digit of Mr. Fibonacci's age be x and the ones digit be y. Then Mr. Fibonacci's age is $10x + y$, whereas Mrs. Fibonacci's age if $10y + x$. Therefore, the difference between their ages is $(10x + y) - (10y + x) = 9x - 9y = 9(x - y)$. Since x and y are digits, the minimal positive value of $x - y$ is 1. Therefore, the smallest possible difference between their ages is $9 \times 1 = 9$.
Note. There are several possible ways to achieve this minimal difference. For instance, Mr. Fibonacci can be 32 and Mrs. Fibonacci can be 23, or Mr. Fibonacci can be 89 and Mrs. Fibonacci can be 98.

32. Let N be the product of the first nine multiples of 19 (i.e. $N = 19 \times 38 \times 57 \times \cdots \times 152 \times 171$). What is the last digit of N?
The last digit of a number is 0 if and only if the number is divisible by 10. Note that 19×2 and 19×5 both appear in N, so N is clearly divisible by 10. Therefore, the answer is 0.

33. If two spheres have radii of 2 and 6, what is the ratio of the volume of the larger sphere to the volume of the smaller sphere?
Let us look at the general problem of computing the ratio of the volumes of spheres with radii a and b. Since the volume V of a sphere with radius r can be obtained using $V = \frac{4}{3}\pi r^3$, we see that the desired ratio equals $\frac{4}{3}\pi a^3 \div \frac{4}{3}\pi b^3 = (\frac{a}{b})^3$. Substituting $a = 6$ and $b = 2$ yields the answer $(\frac{6}{2})^3 = 27$.
Note. This solution can actually be generalized to any two similar, three-dimensional figures. The ratio of the volumes of such figures equals the cube of the ratio of similarity. One can also show that the ratio of the areas of two similar, two-dimensional figures is equal to the square of the ratio of similarity.

34. If a, b, and c are nonzero numbers satisfying $3a = 4b$ and $5b = 6c$, what is $\frac{c}{a+b}$?
First Solution. From the given, we have $a = \frac{4}{3}b$ and $b = \frac{6}{5}c$. Substituting the second into the first, we also have $a = \frac{4}{3} \cdot \frac{6}{5}c = \frac{8}{5}c$. Therefore, $a + b = \frac{8}{5}c + \frac{6}{5}c = \frac{14}{5}c$, so $\frac{c}{a+b} = \frac{5}{14}$.
Second Solution. A cheap way of solving such a problem is as follows. We first let a be any nonzero value; for instance, let $a = 1$. Substituting this into the first given equation yields $3 = 4b$, or $b = \frac{3}{4}$. Now, substituting this into the second given equation yields $3 \times \frac{5}{4} = 6c$, or $c = \frac{5}{8}$. Then, $\frac{c}{a+b} = \frac{5}{8} \div (1 + \frac{3}{4}) = \frac{5}{8} \div \frac{7}{4} = \frac{5}{14}$.
Note. The second solution displays a powerful problem solving technique: substituting numbers. Sometimes, if a problem implies the existence of an answer, simply plug in numbers to *force* the answer without really establishing it.

35. Point X is 210 miles from point Y. Car A starts at X and drives towards Y at 40 mph. Car B starts at Y and drives towards X at 50 mph. If both cars start at noon, at what time will they meet?
Assume that t hours later, the cars meet at point P. This implies that the distance from point X to point P is $40t$ miles, and that the distance from point Y to point P is $50t$ miles. Since the distance between X and Y is 210 miles, we must have $40t + 50t = 210$, or $t = \frac{210}{90} = \frac{7}{3}$ hours. Since $\frac{7}{3}$ hours is equal to $\frac{7}{3} \times 60 = 140$ minutes, or 2 hours and 20 minutes, the cars must meet at 2:20PM.

36. A slice of pizza costs \$1. A pie, which is composed of 8 slices, costs \$6. Bob buys individual slices and pies so that he has 78 total slices. If he has to pay \$62, how many slices does he buy individually?
Assume that Bob bought x individual slices. Since he has a total of 78 slices, he must have bought $\frac{78-x}{8}$ pies. Calculating the price of his purchases, we have the equation $(x \times 1) + (\frac{78-x}{8} \times 6) = 62$, or $x + \frac{3}{4} \cdot (78 - x) = 62$. Multiply both sides by 4 to clear fractions: $4x + 3(78 - x) = 248$. Simplifying yields $4x + 234 - 3x = 248$, or $x = 14$. Therefore, he bought 14 individual slices.

37. If $A + B = \sqrt{8}$ and $A - B = \sqrt{5}$, what is $A \times B$?
First Solution. Square both sides of both equations to obtain $A^2+2AB+B^2 = 8$ and $A^2-2AB+B^2 = 5$. Now subtracting the second one from the first yields $4AB = 3$, or $AB = \frac{3}{4}$.
Second Solution. Adding the two given equations yields $2A = \sqrt{8}+\sqrt{5}$, or $A = \frac{\sqrt{8}+\sqrt{5}}{2}$. Subtracting the second from the first yields $2B = \sqrt{8} - \sqrt{5}$, or $B = \frac{\sqrt{8}-\sqrt{5}}{2}$. Therefore, $AB = \frac{(\sqrt{8}+\sqrt{5})(\sqrt{8}-\sqrt{5})}{4}$. Using the fact that $(x + y)(x - y) = x^2 - y^2$, we see that $(\sqrt{8} + \sqrt{5})(\sqrt{8} - \sqrt{5}) = (\sqrt{8})^2 - (\sqrt{5})^2 = 8 - 5 = 3$, so $AB = \frac{3}{4}$.

38. Let $ABCD$ be a rectangle with $\overline{AB} = 20$ and $\overline{BC} = 6$. Let P be the point such that P is 12 units away from $\overline{DA}$ and 5 units away from $\overline{CD}$. What is the area of quadrilateral $APCD$?
Since triangle APD has area $\frac{1}{2} \times 6 \times 12 = 36$ and triangle DPC has area $\frac{1}{2} \times 20 \times 5 = 50$, the area of quadrilateral $APCD$ is $36 + 50 = 86$.

39. What is the 100th digit after the decimal point when $\frac{1}{7}$ is written in decimal form?
Actually carrying out the division $1 \div 7$, we can compute that $\frac{1}{7} = 0.142857142857142857...$, and note that the digits 142857 continue to repeat itself after the decimal point. Therefore, the 100th digit after the decimal point will be 8.
Note. Every *rational number* (the quotient of two integers) expressed as a decimal repeats indefinitely like the above example.

40. Simplify $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \frac{1}{5 \cdot 6} + \frac{1}{6 \cdot 7}$ in lowest terms.
First Solution. It is easy to establish that $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$. (Simplify the right side using a common denominator.) Therefore, the given expression is equivalent to $(\frac{1}{1} - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + (\frac{1}{4} - \frac{1}{5}) + (\frac{1}{5} - \frac{1}{6}) + (\frac{1}{6} - \frac{1}{7}) = \frac{1}{1} - \frac{1}{7} = \frac{6}{7}$.
Second Solution. We can also calculate the sum directly, though this method is more time-consuming. First, we need a common denominator for all of the given fractions, so we must compute the least common multiple of the numbers 1, 2, 3, ..., 7. This number should be divisible by 2 at least 2 times, divisible by 3 at least once, divisible by 5 at least once, and divisible by 7 at least once. Therefore, the desired denominator is $2^2 \times 3 \times 5 \times 7 = 420$. Now, we can evaluate the expression: $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \frac{1}{5 \cdot 6} + \frac{1}{6 \cdot 7} = \frac{210}{420} + \frac{70}{420} + \frac{35}{420} + \frac{21}{420} + \frac{14}{420} + \frac{10}{420} = \frac{360}{420} = \frac{6}{7}$.

41. A circle has a diameter of 376,216 inches. Using the approximation $\pi = 3.14$, compute the ratio of the circle's area, in square inches, to its circumference, in inches.
The area A and the circumference c of a circle with radius r can be expressed by the formulas $A = \pi r^2$ and $c = 2\pi r$, so the ratio of the area to the circumference is equal to $\frac{A}{c} = \frac{\pi r^2}{2\pi r} = \frac{r}{2}$. Since $2r = 376,216$, we have $r = 188,108$, and substituting this into the aforementioned expression, we see that the answer is $\frac{r}{2} = \frac{188108}{2} = 94,054$.
Note. The given approximation is useless, since π cancels out in the desired ratio.

42. In isosceles triangle ABC, $\overline{AB} = \overline{AC}$ and $\angle BAC = 140^\circ$. Point D lies on $\overline{BC}$ such that $\overline{AD} = \overline{BD}$. Compute $\angle DAC$ in degrees.
Since the sum of the angles in triangle ABC is 180° and $\angle ABC = \angle ACB$, we must have $\angle ABC = \angle ACB = 20^\circ$. Because triangle ADB is isosceles, $\angle DAB = \angle DBA = 20^\circ$, and $\angle DAC = \angle BAC - \angle DAB = 140^\circ - 20^\circ = 120^\circ$.

43. What are the last 2 digits of 5^{2003}?
If N is any integer whose final two digits are 25, then $N = 100x + 25$, so $5N = 5(100x + 25) = 100(5x) + 125 = 100(5x + 1) + 25$. The $100(5x + 1)$ term obviously cannot contribute to the last 2 digits of $5N$, which implies that the last two digits of $5N$ must be 25 as well. In other words, if N ends in 25, then $5N$ ends in 25. Since $5^2 = 25$ ends in 25, 5^3 ends in 25, and thus 5^4 ends in 25 also. Continuing this logic, we see that 5^{2003} must end in 25 as well.

44. Which of the following are true, given that there may be more than one true statement:
I: the square of an integer is a whole number
II: if $a \neq b$ and $b \neq c$, then $a \neq c$
III: every integer has a rational inverse in multiplication
IV: the square root of a positive integer is real
Statements I and IV are true. A counterexample for statement II can be obtained by letting $a = 1$, $b = 2$, and $c = 1$. A counterexample (in fact, the only counterexample) for statement III is zero, which does not have an inverse in multiplication.

45. A number is *strictly decreasing* if each digit is strictly less than the digit to its left. For example, 531 and 962 are *strictly decreasing*, whereas 562 and 322 are not. How many integers between 100 and 600 are *strictly decreasing*?

It is easy to simply list all of them: 210, 310, 320, 321, 410, 420, 421, 430, 431, 432, 510, 520, 521, 530, 531, 532, 540, 541, 542, 543. Therefore, there are 20.

46. What is the units digit of $13^{17} + 17^{13}$?
Note that in calculating the units digit of a sum or product, the other digits are irrelevant. In other words, 13^{17} and 3^{17} have the same last digit. Now, note that $3^2 = 9$ has last digit 9, and therefore, the last digit of $3^3 = 3^2 \times 3$ is the same as the last digit of 9×3, or 7. Furthermore, the last digit of $3^4 = 3^3 \times 3$ is the same as the last digit of 7×3, or 1. Continuing this process, we note that the last digit of $3^5 = 3^4 \times 3$ is the same as the last digit of 1×3, or 3, and eventually, we see that there is a cycle of period 4. In other words, the last digits of 3^1, 3^2, 3^3, 3^4, 3^5, 3^6, 3^7, 3^8, 3^9, $\ldots$, form the cycle 3, 9, 7, 1, 3, 9, 7, 1, 3, and etc. Therefore, the last digit of 13^{17} is 3. Similarly, the last digits of 7^1, 7^2, 7^3, 7^4, 7^5, 7^6, 7^7, 7^8, 7^9, $\ldots$, form the cycle 7, 9, 3, 1, 7, 9, 3, 1, 7, and etc. We thus deduce that the last digit of 17^{13} is 7. Consequently, the last digit of $13^{17} + 17^{13}$ is the same as the last digit of $3 + 7$, which is 0.

47. There are ten lottery tickets in a hat, and four of them are winning tickets. First, Joe reaches in and takes a ticket. Then, Kim reaches in and takes a ticket from the remaining nine. What is the probability that Kim takes a winning ticket?
There are two independent cases to consider: Either Joe takes a winning ticket, or he takes a losing ticket. The probability that Joe takes a winning ticket and Kim takes a winning ticket is $\frac{4}{10} \times \frac{3}{9} = \frac{12}{90} = \frac{2}{15}$. The probability that Joe takes a losing ticket and Kim takes a winning ticket is $\frac{6}{10} \times \frac{4}{9} = \frac{24}{90} = \frac{4}{15}$. Thus, the total probability that Kim takes a winning ticket is $\frac{2}{15} + \frac{4}{15} = \frac{6}{15} = \frac{2}{5}$.
Note. Amazingly enough, the probability that Kim wins is exactly the same as the probability that she wins when she is the first to choose a ticket. This is not a coincidence, however, and in such a game of choosing lottery tickets without replacement, it can be proved that the order in which the people choose is irrelevant.

48. A *silly* number *ababab* is formed by repeating a two-digit number *ab* exactly three times. For example, 252525 is a *silly* number. What is the greatest common divisor of all *silly* numbers?
Let N denote the two-digit number *ab*. Then, the number *ababab* is equal to $10000N + 100N + N = 10101N$. In other words, all *silly* numbers are of the form $10101N$, where N is an arbitrary two-digit number. Since the greatest common factor of all possible values of N is obviously 1, (For instance, N can be 11 and 12, which are relatively prime.) the greatest common factor of all *silly* numbers is 10101.

49. A number p yields a remainder of 3 when divided by 5, a remainder of 5 when divided by 7, and a remainder of 11 when divided by 13. If p is less than 1000, what is the maximum value of p?
The given implies that $p + 2$ is divisible by 5, 7, and 13. Therefore, $p + 2$ must be a multiple of $5 \times 7 \times 13 = 455$. Since p is less than 1000, $p+2$ is less than 1002, so the largest possible value of $p+2$ is $455 \times 2 = 910$. We thus obtain the answer 908.

50. In the *magic square* shown, numbers are to be placed in the empty boxes so that the sums of the numbers in each row, column, and diagonal are equal to the same value. What is the value of x?

19	a	b
	c	26
	x	d

The sum of the first row equals the sum of the last column. We thus have $19 + a + b = b + 26 + d$, so $d - a = -7$. The sum of the second column equals the sum of a diagonal. Therefore, $a+c+x = 19+c+d$, which simplifies to $x = 19 + d - a = 19 + (-7) = 12$.