8^{th} Grade Competition and Solutions

Bergen County Academies Math Competition 2006

22 October 2006

1. How many perfect squares are there between 2 and 140?
   Answer: 10
   The perfect squares between 2 and 140 are $2^2 = 4, 3^2 = 9, \ldots, 11^2 = 131$. So there are $11-2+1 = 10$.

2. When the band The Beatles first formed, they had 1 fan. This fan told 3 other people about The Beatles. Each of these 3 people told 3 other people about The Beatles. As a result, how many people in total knew about The Beatles?
   Answer: 13
   First one person knew, then an additional three people knew, then for each of these three people, an additional three people knew: $1 + 3 + 3 \cdot 3 = 13$.

3. Find the product of the even multiples of 5 that are greater than 1 and less than 49.
   Answer: 240000
   The even multiples of 5 between 1 and 49 are simply the multiples of 10: 10, 20, 30, and 40. $10 \cdot 20 \cdot 30 \cdot 40 = 240,000$.

4. Donald has twice as many books as Kevin. Kevin has 4 times as many books as Watson. Watson has 6 books. How many books do Donald, Kevin, and Watson have in total?
   Answer: 78
   Watson has 6 books, so Kevin has $4 \cdot 6 = 24$ books and Donald has $2 \cdot 24 = 48$ books. In total they have $6 + 24 + 48 = 78$ books.

5. Compute $(1 + 2 + 3 + \cdots + 98 + 99) \div (.01 + .02 + .03 + \cdots + .98 + .99)$.
   Answer: 100
   Note that multiplying each term in the divisor by 100 yields $1 + 2 + 3 + \cdots + 98 + 99$. So the dividend is 100 times the divisor; thus the quotient is 100.

6. The sum of the ages of Sam and Cara is 30. Sam is 4 years younger than Cara. How old is Cara?
   Answer: 17
   Guess-and-check solves this problem fairly quickly. Alternately, let x be Sam's age and y be Cara's age, in years. From the problem, $x + y = 30$ and $y - 4 = x$. Substituting $x = y - 4$ into the first equation yields $y - 4 + y = 30 \Rightarrow 2y - 4 = 30 \Rightarrow 2y = 34 \Rightarrow y = 17$.

7. Farmer John wants to buy fencing for the perimeter of his rectangular field. The length of the field is 17 meters and the area of the field is 119 square meters. Given that fencing costs 10 cents per meter, how much money will Farmer John have to spend?
   Answer: $4.80
   Since the area of a rectangle equals its length times its width, its width equals its area divided by its length. Thus the width of Farmer John's field is $119 \div 17 = 7$ meters. The perimeter of

the field is then $17+17+7+7=48$ meters. So Farmer John must buy 48 meters of fencing at 10 cents per meter, for a total cost of $48 \cdot 10 = 480$ cents, or \$4.80.

8. If a box of donuts is marked down 20$fromitsoriginalpriceof$5, thenmarkedbackup20$ from the sale price, how much does this box of donuts cost now?
   Answer: \$4.80
   Taking 20% off is the same as multiplying by $1 - 0.20 = 0.80$, so a 20% reduction off \$5 brings the price to $0.80 \cdot \$5 = \$4$. Marking this up by 20% is the same as multiplying by 1.20, giving the desired result $1.20 \cdot \$4 = \$4.80$.

9. Find the probability that a randomly chosen positive whole number less than or equal to 90 is a multiple of 17.
   Answer: $\frac{1}{18}$
   The positive multiples of 17 less than or equal to 90 are 17, 34, 51, 68, and 85. There are 5 such numbers out of a total 90. Thus the answer is $\frac{5}{90}$, or $\frac{1}{18}$.

10. Vincent likes to mix different types of soda. He buys one liter of cola for \$2.00, one liter of lemon-lime soda for \$1.50, and 2 liters of orange soda for a total of \$3.50. If he mixes these all together, what is the price per liter of the mixture?
    Answer: \$1.75
    The total cost is $\$2.00 + \$1.50 + \$3.50 = \$7.00$. There are four liters of the mixture. So the price per liter is $\$7.00 \div 4 = \$1.75$.

11. Connie took a test with 25 questions. For every question she got right, she earned 4 points, and for every question she got wrong, she lost 1 point. She answered every question, and got a score of 80. How many questions did she get right?
    Answer: 21
    Guess-and-check can be used to solve this problem. Alternately, let c be the number of questions Connie got right. Then she got $25-c$ questions wrong. So her score is $4c-(25-c) = 5c-25 = 80$. Solve for c: $5c = 105 \Rightarrow c = 21$.

12. Christine can sing 120 notes every minute, while Rachel can sing 180 notes every minute. If Christine and Rachel begin singing at the same time, how long, *in seconds*, will it take for their combined note total to equal 1080 notes?
    Answer: 216
    Together they sing $120 + 180 = 300$ notes every minute. So it takes 1080 notes $\cdot \frac{1 minute}{300 notes} = 3.6$ minutes for them to sing 1080 notes. Convert this to seconds: 3.6 minutes $\cdot\, 60 \frac{seconds}{minute} = 216$ seconds. This uses dimensional analysis: multiplying a given quantity by a ratio so that its unit cancels out and you are left with a different unit.

13. Sally has 7 coins, which together make 72 cents. Each coin is a penny, nickel, dime or quarter. How many dimes does Sally have?
    Answer: 1
    Sally must have 2 pennies, since every other coin has a cent value that is a multiple of 5. If her other 5 coins are all dimes, she would only have 52 cents, so she must have at least one quarter. If her other 4 coins are all dimes, she would only have $25 + 40 + 2 = 67$ cents, so she must have at least two quarters. If her other 3 coins are all dimes, she would have $50 + 30 + 2 = 82$ cents, which is too much, so at least one coin must be a nickel. This would yield $50 + 20 + 5 + 2 = 77$ cents, which is still too much, so two coins must be nickels. Thus she has 2 pennies, 2 quarters, 2 nickels, and 1 dime ($2 \cdot 1 + 2 \cdot 25 + 2 \cdot 5 + 1 \cdot 10 = 72$).

14. What is the number halfway between $\frac{1}{13}$ and $\frac{1}{9}$?
Answer: $\frac{11}{117}$
Find the arithmetic mean of the two fractions. Add the two fractions: $\frac{1}{9} + \frac{1}{13} = \frac{13+9}{9\cdot 13} = \frac{22}{117}$. Divide this sum by 2 to yield $\frac{11}{117}$.

15. On planet Penev, the symbol $\left\{\text{-}\right\}$ is defined as follows: $\left\{\frac{x}{y}\right\} = (-1)^{x+y}$. Compute $\left\{\frac{2}{4}\right\}$.
Answer: 1
From the given definition, $\left\{\frac{2}{4}\right\} = (-1)^{2+4} = (-1)^6 = 1$, since 6 is even.

16. An ant begins at the origin (the point $(0, 0)$) of a Cartesian plane, facing in the positive x-direction. The ant takes two steps forward, turns 90° to its right, takes three steps, turns around (180°), takes five steps, and finally turns 90° to its left and takes nine steps. Assuming each of its steps is the same length, in which quadrant does the ant end up?
Answer: II or 2
Since each step is the same length, let one unit be one step. After its first two steps, the ant is at the point (2,0). After turning right and taking three steps, it is at (2, -3). After turning around and taking five steps, it is at (2, 2). After turning left and taking nine steps, it ends up at (-7, 2), which is in quadrant II.

17. What is the positive difference between the sum of the first fifteen positive even numbers and the sum of the first fifteen positive odd numbers?
Answer: 15
The first fifteen positive even numbers are $2, 4, \ldots, 28$ and the first fifteen odd positive numbers are $1, 3, \ldots, 27$. The desired difference is then $(2 + 4 + \cdots + 28) - (1 + 3 + \cdots + 27)$. By associativity, this equals $(2 - 1) + (4 - 3) + \cdots + (28 - 17) = 1 + 1 + \cdots + 1$, where there are fifteen 1's in the final expression. So the difference equals 15.

18. The point (10, 0) is 10 units from the origin. Give a point, as an ordered pair (x, y), that is also 10 units from the origin such that both coordinates x and y are positive whole numbers and $x < y$.
Answer: (6, 8)
Since $(x,\ y)$ is 10 units from the origin, using the distance formula (or noting that the line from $(x,\ y)$ to the origin can be viewed as the hypotenuse of a right triangle) we have $x^2 + y^2 = 10^2$. The only positive whole numbers $(x,\ y)$ satisfying this (with $x < y$) are $x = 6$, $y = 8$.

19. Let $N = 20 \cdot 30 \cdot 50 \cdot 70 \cdot 90 \cdot 110 \cdot 130$. What is the smallest prime number that is not a factor of N?
Answer: 17
From the expression given for N in the problem, it can be seen that $N = 2\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11\cdot 13\cdot 10^7$. So the first six prime numbers $2, 3, 5, 7, 11$ and 13 are all factors of N, while 17 - the next prime - is not.

20. Compute $\left(\sqrt{2^3}\right)^{\frac{4}{3}}$.
Answer: 4
Taking a square root is equivalent to raising to the $\frac{1}{2}$ power: $\sqrt{2^3} = (2^3)^{\frac{1}{2}}$. Using the rules of exponents, we see that $\left((2^3)^{\frac{1}{2}}\right)^{\frac{4}{3}} = 2^{3\cdot\frac{1}{2}\cdot 4\cdot\frac{1}{3}} = 2^2 = 4$.

21. If the length, width, and height of a box (rectangular prism) are all doubled, by how many times is the volume of the box increased?
Answer: 8

Since volume = length $\cdot$ width $\cdot$ height , the increased volume is $2l \cdot 2w \cdot 2h = 8lwh = 8v$, where l, w, h and v represent the original length, width, height, and volume, respectively.

22. In the game of Gaussball, there are two ways to score points: a regular goal, worth 3 points; and a Gauss goal, worth 4 points. What is the greatest whole number that cannot be earned as a score in a game of Gaussball?
Answer: 5
A score of 6 is possible by scoring two regular goals. One regular goal and one Gauss goal earn a score of 7, and two Gauss goals earn a score of 8. Any greater whole number can be earned as a score by earning $6, 7$ or 8 points and then scoring some number of regular goals. So every whole number greater than or equal to 6 can be earned as a score in Gaussball, while 5 cannot.

23. Eddy scored 4 free throws out of the 13 free throws he has taken. How many consecutive shots must he now score in order to achieve a shooting percentage (free throws scored out of total free throws taken) of 50%?
Answer: 5
Add one more free throw scored and one more free throw taken until the number taken is twice the number scored: $\frac{5}{14}, \frac{6}{15}, \frac{7}{16}, \frac{8}{17}, \frac{9}{18}$. 9 is half of 18, so if he scores the next $9 - 4 = 5$ consecutive free throws, he will have a shooting percentage of 50%.
Alternately, let x be the number of consecutive shots he scores after the first 13 free throws. We want $\frac{4+x}{13+x} = \frac{1}{2}$. Solve for x: $2(4 + x) = 13 + x \Rightarrow 8 + 2x = 13 + x \Rightarrow x = 5$.

24. Three ducks - Duck 1, Duck 2, and Duck 3 - are sitting in a row. Duck 1 quacks once every six minutes. Duck 2 quacks once every nine minutes. Duck 3 quacks once every fifteen minutes. Given that all three ducks quack at 1:00 PM, when is the next time they will all simultaneously quack? ("Simultaneously" means at the same time.)
Answer: 2:30 PM
Find the least common multiple of the times between quacks: the least common multiple of 6, 9, and 15 is 90 $(2 \cdot 3 \cdot 3 \cdot 5)$. So all three ducks quack simultaneously every 90 minutes. Since they all quack at 1:00 PM, the next time they all quack is 90 minutes later, or 2:30 PM.

25. How many lines of symmetry does a regular dodecagon (a polygon with 12 sides) have?
Answer: 12
Since 12 is even, the lines of symmetry will either join two opposite vertices through the center, or join the midpoints of 2 opposite sides through the center. There are 6 of the first type and 6 of the second type, making 12 total.

26. To complete his meal, James must select 3 different side dishes and one drink. There are 6 available different side dishes and 4 available drinks. The order in which James selects the side dishes and drinks does not matter. How many different meals can James select?
Answer: 80
James can select $\binom{6}{3} = \frac{6!}{3!3!} = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = 20$ different combinations of side dishes. For each of these combinations of side dishes, he has four choices of drink, for a total of $4 \cdot 20 = 80$ different meals.
Note: $\binom{n}{k}$, read "n choose k," denotes the expression for the number of ways to choose k elements from a set of n elements. It is equal to $\frac{n!}{k!(n-k)!}$.

27. Compute $1001^2 - 999^2$.
Answer: 4000
The two squares could be computed and the difference could be found directly. It is quicker to recall the difference of two squares factorization: $a^2 - b^2 = (a + b)(a - b)$. Applying this to the given problem, we have $1001^2 - 999^2 = (1001 + 999)(1001 - 999) = 2000 \cdot 2 = 4000$.

28. The angle measures of *acute* triangle ABC are all positive whole numbers. If $m\angle A = 50°$, what is the smallest possible measure, in degrees, of $\angle B$?
Answer: 41
The sum of the three internal angles of a triangle is 180°. So $m\angle B + m\angle C = 180° - m\angle A = 180° - 50° = 130°$. The measure of $\angle B$ is smallest when $\angle C$ is largest. Since $\triangle ABC$ is acute, each angle measures less than 90°, so the maximum value for the measure of $\angle C$ is 89°. Thus the smallest value for the measure of $\angle B$ is $130° - 89° = 41°$.

29. What is the units digit of 8^{22}?
Answer: 4
When finding the units digits of successive powers of 8, it is only necessary to multiply the preceding units digit by 8. $8^1 = 8$ and $8^2 = 64$, so the units digit of 8^3 is the units digit of $4 \cdot 8 = 32$, which is 2. The units digit of 8^4 is then 6, and the units digit of 8^5 is 8. Since we have only been multiplying by 8 and now the units digit 8 has repeated, we have a repeating pattern of units digits: 8, 4, 2, 6. Since this pattern consists of 4 terms, and 20 is a multiple of 4, the units digit of 8^{22} is the same as the units digit of 8^2, which is 4.

30. Dan runs at a speed of 1.1π miles per hour. How long, in hours, does it take Dan to run 2.75 times around a circle with radius 1 mile?
Answer: 5
The circumference of a circle equals $2\pi r$, where r is the circle's radius. So the circle in the problem has circumference $2\pi \cdot 1 = 2\pi$ miles. Running around it 2.75 times is thus $2.75 \cdot 2\pi = 5.5\pi$ miles. Since distance=rate·time, time=distance÷rate and Dan's time is 5.5π miles $\div$ 1.1π miles/hour $= 5$ hours.

31. Dr. Mayers raises chickens and rabbits in his classroom. He counts 450 total animals, and 1050 total legs on the animals. Each chicken has 2 legs, while each rabbit has 4 legs. How many rabbits does Dr. Mayers have?
Answer: 75
Let the number of chickens be c and the number of rabbits be r. From the given information, $r + c = 450$ and $4r + 2c = 1050$. Multiplying the first equation by 2 and subtracting it from the second yields $2r = 150$. Therefore $r = 75$.
Note: This problem can also be solved without using any algebra. The classic method is to first assume that all the animals are chickens, and then switch in a rabbit for a chicken until the leg number becomes 1050.

32. A *palindrome* is a number that reads the same left-to-right as right-to-left. For example, 1234321 is a palindrome. How many palindromes between 1000 and 9999 are divisible by 9?
Answer: 10
Note that the sum of the digits of any multiple of 9 is divisible by 9. A palindrome between 1000 and 9999 must be of the form $abba$, where a and b are digits and $a \neq 0$. So $a+a+b+b = 2(a+b)$ must be divisible by 9, which means that $a + b$ is divisible by 9. Then the possibilities for the ordered pair (a, b) are $(1, 8), (2, 7), (3, 6), (4, 5), (5, 4), (6, 3), (7, 2), (8, 1), (9, 0)$, and $(9, 9)$, for a total of 10 palindromes.

33. A cylindrical container of internal radius 2 inches and internal volume 32π cubic inches is $\frac{3}{4}$ filled with water. What is the maximum number of cylindrical disks of radius 2 inches and height $\frac{3}{5}$ inch that can be added to the container without letting the water overflow?
Answer: 3
The radius of each disk is the same as the radius of the container, so we are only concerned with the heights. The container is a cylinder, so the formula $V = \pi r^2 h$ applies; solving for h gives a

height of 8 inches. Because it is $\frac{3}{4}$ filled, the water is at a height of $\frac{3}{4} \cdot 8 = 6$ inches. Adding 3 disks will increase the water level to 7.8 inches, but adding 4 will cause the water level to exceed 8 inches, overflowing the container.

34. If $2^x \cdot 3^x + 6^x = 432$, find x.
Answer: 3
Note that $2^x \cdot 3^x = (2 \cdot 3)^x = 6^x$, so the given expression is equivalent to $2 \cdot 6^x = 432 \Rightarrow 6^x = 216 \Rightarrow x = 3$.

35. On a test, Jamie was asked to add 9 to a number x, and then divide the result by 2. However, Jamie accidentally *subtracted* 9 from x and then *multiplied* the result by 2. Luckily for her, she still got the correct answer! Find x.
Answer: 15
$(x + 9) \div 2$ and $(x - 9) \cdot 2$ must be the same number. Set the two expressions equal and solve for x: $\frac{x+9}{2} = 2(x - 9) \Rightarrow x + 9 = 4x - 36 \Rightarrow 45 = 3x \Rightarrow x = 15$.

36. Richard plays darts on a circular dartboard that is made up of two concentric circles (two circles with the same center but different sizes). Getting a bulls-eye means hitting anywhere in the inner circle, which has radius 2 inches. The outer circle has radius 3 inches. Richard always hits the dartboard, but does not always get a bulls-eye. If Richard has an equal chance of hitting each point on the dartboard, what is the probability that Richard gets a bulls-eye in his next shot?
Answer: $\frac{4}{9}$
The area of a circle is πr^2, where r is the circle's radius. Since the probability of hitting each point is equal, divide the bulls-eye area by the total area: $\frac{\pi \cdot 2^2}{\pi \cdot 3^2} = \frac{4}{9}$.

37. Four different numbers A, B, C and D are chosen from the set

$$\{-4, -\frac{3}{2}, -1, 0, \frac{5}{8}, 1, 4, 7\}.$$

What is the smallest possible value of the sum $\frac{A}{B} + \frac{C}{D}$?
Answer: $-\frac{67}{5}$
The sum is smallest when both fractions are negative and as far from zero as possible. This occurs when the two fractions are $\frac{7}{-1}$ and $\frac{-4}{5/8}$, which sum to $-\frac{67}{5}$.

38. Aaron and John have a 300-square-foot lawn. It takes Aaron 30 minutes to mow the lawn alone, and it takes John 20 minutes to mow the lawn alone. If Aaron and John work together to mow the lawn at the same time, how long, in minutes, will it take them to mow it?
Answer: 12
Aaron mows at a rate of $\frac{300sqft}{30min} = 10\frac{sqft}{min}$, and John mows at a rate of $\frac{300sqft}{20min} = 15\frac{sqft}{min}$. Together they mow at a rate of $10+15 = 25\frac{sqft}{min}$. So they mow the 300-square-foot lawn in $\frac{300sqft}{25sqft/min} = 12$ minutes.

39. If $ax = bx$ and $a \neq b$, compute $7(a + 6b)^x$.
Answer: 7
The only x satisfying $ax = bx$ and $a \neq b$ is $x = 0$. Any number raised to the 0^{th} power equals 1, so $(a + 6b)^x = (a + 6b)^0 = 1$ and $7(a + 6b)^x = 7 \cdot 1 = 7$.

40. Dr. Crane gave her class five homework assignments. Arthur only did three of these assignments. Dr. Crane picks two assignments at random to check, out of the five she gave. What is the probability that Arthur has done both of the assignments that Dr. Crane checks?

Answer: $\frac{3}{10}$
For the first assignment that Dr. Crane checks, there is a $\frac{3}{5}$ chance that Arthur has done it. If Arthur has done this assignment, then there are four remaining assignments Dr. Crane could check, and Arthur has done two of them, so there is a $\frac{3}{5} \cdot \frac{2}{4} = \frac{6}{20} = \frac{3}{10}$ chance Arthur has done both assignments.

41. Josh wonders who else participates in math team, physics team, and choir. There are 300 people who participate in at least one of the three activities, 150 who participate in choir, 200 who participate in math team, 32 who participate in physics team, 70 who participate in both choir and math team, 12 who participate in both choir and physics team, and 8 who participate in both math team and physics team. How many people participate in all three activities?
Answer: 8
Note that (total number of people) - (number of people who participate in one activity) + (number of people who participate in two activities) - (number of people who participate in all three activities) $= 0$. So the number of people who participate in all three activities is $(300) - (150 + 200 + 32) + (70 + 12 + 8) = 8$. This is an example of the principle of inclusion-exclusion. Drawing a Venn diagram may help solve this problem.

42. Kite $MATH$ has right angles at A and H, and $m\angle M = \theta$. Extend lines MA and TH so they intersect at point K, and extend lines MH and AT so they intersect at point L. Finally, draw line KL. What is $m\angle MLK$, in terms of θ?
Answer: $90^\circ - \frac{\theta}{2}$
Since kite $MATH$ is symmetric over line MT, line segments MK and ML are the same length and $m\angle MKL = m\angle MLK$, so KLM is an isosceles triangle. Because $\angle KML$ is the vertical angle of $\angle HMA$, it has the same measure, so $m\angle KML = \theta$. Then the sum of angles MLK and MKL is $180^\circ - \theta$. Since the two angles are equal, $m\angle MLK$ is half of $180^\circ - \theta$, or $90^\circ - \frac{\theta}{2}$.

43. Jimbo, Jun, Jay, Jacob, Janine and Jenna are standing in a line with their backs to a wall. How many different orders could they be standing in so that Jenna and Jimbo are standing next to each other?
Answer: 240
Since Jenna and Jimbo must be next to each other, we can treat them for the moment as one unit. Thus we have five units to line up, which can be done in $5! = 5\cdot4\cdot3\cdot2\cdot1 = 120$ ways. Since Jenna and Jimbo can switch places with each other, multiply this number by two: $2 \cdot 120 = 240$.

44. How many positive divisors does $(((2^0 \cdot 3^1)^2 \cdot 4)^3 \cdot 5)^4$ have?
Answer: 3125
Expanding the exponentiation, the number is $3^{24} \cdot 4^{12} \cdot 5^4 = 2^{24} \cdot 3^{24} \cdot 5^4$. Each divisor will take the form $2^a 3^b 5^c$, where $0 \le a \le 24, 0 \le b \le 24, 0 \le c \le 4$. The values of a, b, c can be any whole number in these intervals, so there are $25 \cdot 25 \cdot 5 = 3125$ divisors.

45. Each term in the Almost-Fibonacci sequence (except for the first two terms) is equal to the sum of the previous two terms. That is, $a_n = a_{n-1} + a_{n-2}$ for all $n \ge 3$, where a_n is the n^{th} term in the sequence. If the second term in the sequence is 4 and the ninth term is 110, what is the first term in the sequence?
Answer: 2
Let the first term in the sequence be a. Since the second term is 4, the third term is $a + 4$. Then the fourth term is $a + 8$, the fifth term is $a + 4 + a + 8 = 2a + 12$, the sixth term is $a + 8 + 2a + 12 = 3a + 20$, the seventh term is $2a + 12 + 3a + 20 = 5a + 32$, the eighth term is $3a + 20 + 5a + 32 = 8a + 52$, and the ninth term is $5a + 32 + 8a + 52 = 13a + 84$, which we know equals 110. Solve for a: $13a + 84 = 110 \Rightarrow a = 2$.

46. Ben and Joe are playing a game. They flip coins one at a time until either the sequence heads-tails or tails-tails comes up. If the sequence is heads-tails, Ben wins. If the sequence is tails-tails, Joe wins. What is the probability that Ben wins?
Answer: $\frac{3}{4}$
Suppose the first coin comes up heads. This occurs with probability $\frac{1}{2}$. Then Ben must win: even if they continue flipping heads, a tails must eventually come up, at which point Ben wins. Suppose the first coin comes up tails. If the next coin comes up tails, Joe wins. This happens with probability $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. If the next coin comes up heads, Ben wins, as explained above. This happens with probability $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. So the total probability that Ben wins is $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$.

47. What is the largest value of n such that $n^2 - 6n - 55$ is prime?
Answer: 12
Let $P = n^2 - 6n - 55$ and notice that $n^2 + 6n - 55 = (n+5)(n-11)$. Unless $n+5$ or $n-11$ equals 1, this represents a factorization of P, making P non-prime. In order for P to be prime, either $n + 5 = 1 \Rightarrow n = -4$ or $n - 11 = 1 \Rightarrow n = 12$. Take the larger value: $n = 12$. (Indeed, $12^2 - 6 \cdot 12 - 55 = 17$ is prime.)

48. A rectangle is divided into congruent squares. The sum of the perimeters of the squares is numerically equal to the area of the rectangle. What is the length of a side of one of the squares?
Answer: 4
If the rectangle is divided up so that there are m divisions along the height and n along the width yielding squares with side length a, the area of the rectangle is $mn \cdot a^2$, since there are mn squares of area a^2. The total perimeter of these squares is $mn \cdot (2a + 2a) = 4mna$. Thus $4mna = mna^2$, so, canceling out, $a = 4$.

49. A mouse starts on the coordinate plane at the origin (the point $(0, 0)$). Each move, the mouse randomly chooses one of the following four directions and moves one unit in that direction: up (positive y-direction), down (negative y-direction), left (negative x-direction), or right (positive x-direction). In how many ways can the mouse end at the point $(2, 0)$ after 25 such moves?
Answer: 0
In order to start at $(0, 0)$ and end up at $(2, 0)$, the mouse must make the same amount of moves upwards as moves downwards. So the total number of moves upwards and downwards must sum to an even number. Since the mouse must end up 2 units to the right of where it started, it must make two more movements to the right than to the left. This means that the number of moves left and the number of moves right must both be odd or both be even, which means that the total number of movements left and right sums to an even number. Since the number of up and down movements total an even number, and the number of right and left movements total an even number, in order to end up at (2,0), the mouse must have make an even amount of moves. The mouse makes 25 moves (an odd number) in total, and therefore it cannot end at $(2, 0)$.

50. Andrei challenges the math team to a math duel. For each question in the duel, Andrei must give exactly one answer. Each member of the math team may also give one answer, and if at least one math team member gives the correct answer, the math team gets the question right. Andrei has a $\frac{31}{32}$ chance of answering any given question correctly, while each member of the rest of the math team has a $\frac{1}{2}$ chance of getting any given question correct. Find the number of math team members that should answer a question so that the math team has exactly the same probability of answering the question correctly as Andrei does. (Note: Andrei is not considered a member of the math team.)
Answer: 5

Find the probability of the complement event: the probability that at least one math team member answers the question correctly equals 1 minus the probability that no math team member answers it correctly. Let N be the number of math team members. The probability that all N math team members get the question wrong is $\left(\frac{1}{2}\right)^N$. So $1 - \left(\frac{1}{2}\right)^N = \frac{31}{32}$. Solving for N yields $N = 5$.