5th Grade Competition Solutions

Bergen County Academies Math Competition
19 October 2008

1. Before taking the AIME, a student notices that he has a pack of Smarties, two bags of chips, and a Hershey's kiss on his desk. Each pack of Smarties has 50 calories, each bag of chips has 120 calories, and each Hershey's kiss has 10 calories. If the student eats all of the food on his desk during his test, then how many calories will he have eaten?
Answer: 300
$1\times 50+2\times 120+1\times 10 = 50+240+10 = 300$.

2. $12+34-56-78+90 = ?$
Answer: 2
$12+34-56-78+90 = 46-56-78+90 = -10-78+90 = -88+90 = 2$.

3. Pavel is driving at 75 miles per hour. Looking into his rearview mirror, he notices that he is traveling 17 miles per hour faster than the truck behind him. At what speed is the truck traveling, in miles per hour?
Answer: 58
The truck is traveling $75-17 = 58$ miles per hour.

4. Nikhil says the word 'horse' ten times per hour. If he is awake for 16 hours per day, then how many times will he say the word 'horse' in a 24-hour period? (Nikhil does not talk in his sleep.)
Answer: 160
For 16 hours of the day, Nikhil will say 'horse' ten times per hour. Thus, in any 24-hour period, Nikhil will say 'horse' $16\times 10 = 160$ times.

5. Find the perimeter in inches of a square that has area 49 ft^2.
Answer: 336
The square has side length equal to 7 feet, and thus a perimeter of 28 feet. Since there are 12 inches in a foot, 28 feet $= 28\times 12 = 336$ inches.

6. $(300+50+8)\times(1000+20+4) = ?$
Answer: 366592
$(300+50+8)\times(1000+20+4) = 358\times 1024 = 366592$.

7. Right triangle ABC has a right angle at C and side lengths $\overline{AC} = 6$ and $\overline{BC} = 3$. Compute the area of $\triangle ABC$.
Answer: 9
The area of a right triangle with legs a and b is $\frac{1}{2}ab$. In this case, that is $\frac{1}{2}\times 6\times 3 = 3\times 3 = 9$.

8. If a ball is moving with a constant speed of 5 meters per second over a duration of 24 seconds, how many meters does the ball move?
Answer: 120
The ball moves 5 meters per second, so it moves $24\times 5 = 120$ meters in 24 seconds.

9. Find the largest integer value of a for which the statement "2048 is divisible by 2^a" is true.
Answer: 11
$2^{11} = 2048$.

10. Robert's backyard measures 37 by 53 feet. What is the area of Robert's backyard, in ft^2?
Answer: 1961
The area of the backyard is $37\times 53 = 1961$ ft^2.

11. Nikhil is playing ping pong. In a game to 15, he was losing 10-4. Then, Nikhil scored a more points than his opponent and won the game. What is the smallest possible value of a?
Answer: 7
Nikhil must score exactly 11 points to win. To minimize a, we must maximize the number of points that Nikhil's opponent scores. This happens when his opponent scores 14 points. Thus, the minimum value of a is $11-4=7$.

12. Robert's Pokemon card collection contains 9135 cards, including 289 holographic ones. Robert gives away 568 of his non-holographic cards. How many non-holographic cards does he have now?
Answer: 8278
Before the transaction, Robert had $9135-289=8846$ non-holographic cards. So after the transaction, Robert has $8846-568=8278$ non-holographic cards.

13. Tom has a rectangular field of dimensions 15 x 36 feet. A post must be placed no more than three feet from any other post, and each corner must have a post. What is the minimum number of posts that Tom needs to surround his field?
Answer: 34
Other than the corner posts, the sides of length 36 feet have at least $\frac{36}{3}-1=12-1=11$ posts, and each of the 15 foot sides have at least $\frac{15}{3}-1=5-1=4$ posts. In all, Tom needs at least $4+2\times 11+2\times 4=4+22+8=34$ posts.

14. Compute the hypotenuse of a right triangle that has legs of lengths 3 and 9.
Answer: $3\sqrt{10}$
The Pythagorean theorem says that in a right triangle with legs a and b and hypotenuse c, $a^2+b^2=c^2$. So, the hypotenuse of the triangle is equal to $\sqrt{a^2+b^2}=\sqrt{3^2+9^2}=\sqrt{9+81}=\sqrt{90}=3\sqrt{10}$.

15. In how many ways can 9 be written as a sum of three different positive integers?
Answer: 3
Consider the smallest of these three integers. It must be at most 2 – otherwise, the sum of the three integers would be at least $3+4+5=12$. When the smallest number is 2, the others must be 3 and 4. When the smallest number is 1, the others can be either 3 and 5 or 2 and 6. This makes 3 possibilities.

16. The area of $\triangle ABC$ is 35. If $\overline{BC}=10$, then find the height from A to $\overline{BC}$.
Answer: 7
The area of a triangle with base b and height h is $\frac{1}{2}bh$. We are given that $\frac{1}{2}\times 10\times h=35\Rightarrow 5\times h=35\Rightarrow h=7$.

17. How many ways can Ally, Brett, and Candy be lined up so that Ally is standing next to her best friend Candy?
Answer: 4
Treat Ally and Candy as one unit. So there are two 'units,' which can be lined up in 2! ways. Now Ally can be either on the left or right of Candy, so our answer is $2\times 2!=4$.

18. If it takes Alex two seconds to type a 5-letter word, then how many seconds will it take him to type 317 5-letter words? Ignore the time it takes to type punctuation and spaces.
Answer: 634
It will take Alex $317\times 2=634$ seconds to type 317 5-letter words.

19. In an auditorium, there are 6 rows of lights. Each row has 7 lights in it. If every light switch in the auditorium controls exactly three lights, then how many light switches are in the auditorium?
Answer: 14
There are $6\times 7=42$ lights in the auditorium. If each switch controls exactly three lights, then there are $42\div 3=14$ switches in the auditorium.

20. The dimensions of a pool are 30 feet wide by 100 feet long by 8 feet deep. If water flows in to the pool at a rate of 150 cubic feet per minute, then how many minutes will it take for the pool to be full with water?

Answer: 160
The volume of the pool is $30 \times 100 \times 8 = 3000 \times 8 = 24000$ ft^3, so it will take $24000 \div 150 = 160$ minutes for the pool to be filled with water.

21. What is the area of a square with a diagonal of length 8?
Answer: 32
The two diagonals of the square divide the square into four congruent isosceles right triangles, each with leg length 4. The area of each of these triangles is $\frac{1}{2} \times 4 \times 4 = 8$, so the area of the entire square is $4 \times 8 = 32$.

22. Ian A. practices the French horn. If he practices 30 minutes per day from Monday to Friday and 45 minutes per day on Saturday and Sunday, then how many hours does he take to practice the French horn in 2 weeks?
Answer: 8
During the five weekdays, Ian practices the horn for $5 \times .5 = 2.5$ hours. On the weekends, he practices the horn for $2 \times .75 = 1.5$ hours. Thus, in two weeks, Ian will spend $2 \times (1.5 + 2.5) = 2 \times 4 = 8$ hours practicing the French horn.

23. The surface area of a soccer ball is two square feet. If 20% of the soccer ball is black, then what is the surface area, in square feet, of the white area on the ball?
Answer: 1.6
If 20% of the soccer ball is black, then $100 - 20 = 80\%$ of it is white. So, the area of white on the soccer ball is $2 \times .8 = 1.6$ ft^2.

24. In a sequence on integers, the first term is 1, the second and third terms are 2, the fourth, fifth, and sixth terms are 3, the next four terms are 4, and so on. What is the sum of the first 10 terms?
Answer: 30
In the first 10 terms, there is one 1, two 2's, three 3's, and four 4's. So, the sum of the first 10 terms is $1 \times 1 + 2 \times 2 + 3 \times 3 + 4 \times 4 = 1 + 4 + 9 + 16 = 5 + 9 + 16 = 5 + 25 = 30$.

25. What is the volume of a square pyramid with base side length 5 and height 36?
Answer: 300
The volume of any pyramid with base area b and height h is $\frac{1}{3}bh$. The square base has area $5 \times 5 = 25$, so the volume of the pyramid will be $\frac{1}{3} \times 25 \times 36 = 25 \times 12 = 300$.

26. A group of people are trying to sit themselves in rows. If they sit in rows of two, there is one person left over. If they sit in rows of five, there are four people left over. What is the smallest possible number of people in this group?
Answer: 9
If there were one more person, then the group of people would be able to sit in rows of 2 and 5 without having anyone left over. So, the smallest possible number of people is one less than the first number that is divisible by 2 and 5. That means the number of people is the number one less than 10, or 9.

27. At math team, 5 bags of chips are equal to 2 cans of soda, and 7 boxes of Nerds are equal to 3 bags of chips. How many cans of soda are 105 boxes of Nerds equal to?
Answer: 18
105 boxes of Nerds are equal to $\frac{3}{7} \times 105 = 3 \times 15 = 45$ bags of chips. 45 bags of chips are equal to $\frac{2}{5} \times 45 = 2 \times 9 = 18$ cans of soda. The calculation can be done directly: $105 \times \frac{3}{7} \times \frac{2}{5} = 105 \times \frac{6}{35} = 3 \times 6 = 18$.

28. Find the area of trapezoid $ABCD$ given $\overline{AB} = 9$, $\overline{CD} = 7$, and the distance between parallel line segments $\overline{AB}$ and $\overline{CD}$ is 3.
Answer: 24
The area of a trapezoid with base lengths b_1 and b_2 and height h is $\frac{1}{2}(b_1 + b_2)h$. In this case, that is $\frac{1}{2}(9 + 7) \times 3 = \frac{16}{2} \times 3 = 8 \times 3 = 24$.

29. If Marina can correctly differentiate between Ben and Joe only $\frac{4}{5}$ of the time, what are the chances that she

correctly identifies Joe twice in a row?
Answer: $\frac{16}{25}$
The probability that she identifies Joe correctly each time is $\frac{4}{5}$, so the probability she identifies him correctly twice in a row is $\frac{4}{5} \times \frac{4}{5} = \frac{16}{25}$.

30. If $1-7x = 2x$, then what is x?
Answer: $\frac{1}{9}$
Adding $7x$ to both sides, we find that $1 = 9x$. Now dividing by 9 on both sides, we see that $x = \frac{1}{9}$.

31. Given that $n! = n(n-1)(n-2)\dots 3\cdot 2\cdot 1$, compute $\dfrac{7!+6!}{5!}$.
Answer: 48
$\dfrac{7!+6!}{5!} = \dfrac{7!}{5!} + \dfrac{6!}{5!} = \dfrac{7\times 6\times 5\times 4\times 3\times 2\times 1}{5\times 4\times 3\times 2\times 1} + \dfrac{6!}{5!} = 7\times 6+6 = 42+6 = 48.$

32. A trapezoid has bases with lengths 6 and 9, and area 105. What is the height of the trapezoid?
Answer: 14
Just as before, the area of a trapezoid with base lengths b_1 and b_2 and height h is $\frac{1}{2}(b_1+b_2)h$. Setting $\frac{1}{2}(6+9)h = 105$, we see that $\frac{1}{2}\times 15\times h = 105 \Rightarrow \frac{h}{2} = 7 \Rightarrow h = 14$.

33. What is the distance from the point $(3,2)$ to the point $(0,6)$?
Answer: 5
The distance from (x_1, y_1) to (x_2, y_2) is $\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$ by using right triangles and the Pythagorean theorem. In this case, that is $\sqrt{(3-0)^2+(2-6)^2} = \sqrt{3^2+(-4)^2} = \sqrt{9+16} = \sqrt{25} = 5$.

34. Find the number of factors of 36.
Answer: 9
$36 = 2^2\times 3^2$. Thus, any factor of 36 will be of the form $2^a\times 3^b$ with $0\le a, b\le 2$. So, there are 3 choices for both a and b, and thus $3\times 3 = 9$ factors of 36.

35. How many zeroes does the number $2^7\times 5^4\times 11$ end in?
Answer: 4
Each zero at the end of a number corresponds to a factor of 10. Thus, the number of factors of 10 a number is divisible by is the number of zeroes it ends in. Since $2^7\cdot 5^4\cdot 11 = 10^4\times 2^3\times 11$, it will end in 4 zeroes.

36. $\dfrac{(27^3)^{\frac{5}{9}}}{27} =?$
Answer: 9
$\dfrac{(27^3)^{\frac{5}{9}}}{27} = \dfrac{27^{\frac{5}{3}}}{27} = 27^{\frac{5}{3}-1} = 27^{\frac{2}{3}} = (3^3)^{\frac{2}{3}} = 3^2 = 9.$

37. $f(x)$ is a function such that $f(1) = 1$ and $f(2n+1) = f(n)+1$. For example, $f(3) = f(2\cdot 1+1) = f(1)+1 = 2$. Compute $f(127)$.
Answer: 7
If $f(2^a-1) = k$, then by the given condition, $f(2(2^a-1)+1) = f(2^a-1)+1 \Rightarrow f(2^{a+1}-1) = k+1$. It is given that $f(2^1-1) = 1$. Thus, $f(127) = f(2^7-1) = 6+f(2^1-1) = 6+f(1) = 7$.

38. What is the degree measure of each angle in a regular hexagon?
Answer: 120
By drawing three non-intersecting diagonals of the hexagon, one can see that it is made up of 4 triangles, so the angles in any hexagon sum to $4\times 180 = 720$ degrees. In a regular hexagon, each of the six angles are equal, so each measures $720^o \div 6 = 120^o$.

39. For what value(s) of x does $\dfrac{x^2-4}{x-2}=0$?

Answer: –2

$\dfrac{x^2-4}{x-2}=\dfrac{(x+2)(x-2)}{x-2}=x+2=0 \Rightarrow x=-2.$

40. Three cups of a mixed drink are made by mixing two cups of cranberry juice with one cup of carbonated water. To this mixture, one more cup of cranberry juice and one more cup of carbonated water are added. What percentage of the drink is now carbonated water?

Answer: 40

There are now five cups of the drink, two of which are carbonated water, so the percentage of carbonated water is $2 \div 5 \times 100\% = 40\%$.

41. How many integers less than 50 have exactly 2 distinct prime factors?

Answer: 23

Instead of counting the number of numbers with 2 prime factors, we will count the number of numbers with more or less than 2 prime factors. One number (1) has no prime factors. The numbers 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47 are all prime and thus have exactly one prime factor. There are 15 of them. Also, the powers of primes that are less than 50, which are 4, 8, 16, 32, 9, 27, 25, and 49, have exactly one prime factor also. There are 8 of them. The numbers $2\times3\times5=30$ and $2\times3\times7=42$ are the only ones less than 50 with three distinct prime factors. There are no numbers less than 50 with four or more distinct prime factors. So our answer is $49-1-15-8-2=49-26=23$.

42. Nikhil is writing the page numbers in a book that has 102 pages. After a certain page number, Nikhil notices that he has written the same number of digits as he still has to write in order to finish numbering all 102 pages. What page number is this?

Answer: 54

Nikhil must write $9\times1+90\times2+3\times3=9+180+9=198$ digits since there are 9 one-digit numbers, 90 two-digit numbers, and 3 three-digit numbers from 100 to 102. So, he will be halfway done after writing $198 \div 2 = 99$ page numbers. This happens on a two-digit page number; in fact, it happens on the $\frac{99-9}{2}=45^{th}$ two-digit page number, which is $9+45=54$.

43. Jae's mom wants to buy a pair of pants that costs \$50. When she gets to the register, she finds out that the store is having a 10% discount. On top of this, she presents a %20 coupon to the cashier. How many dollars does Jae's mom end up paying for the pants?

Answer: 36

Because of the discount, the price of the jeans is reduced to $50\times.9=\$45$. Then, after using the coupon, the price of the jeans is reduced to $45\times.8=\$36$.

44. On a football field, there are 15 blades of grass per square inch. If a football field measures 360 feet by 160 feet, then there are n blades of grass on the entire field. Find the leftmost three non-zero digits of n.

Answer: 124

There are $360\times160=57600$ square feet on the field. Also, there are $12^2=144$ square inches in a square foot, so there are $144\times15=2160$ blades of grass per square foot. Finally, there are $57600\times2160=124416000$ blades of grass on the entire field, and the answer is 124.

45. Compute $\imath^{2008}-\imath^{2006}$, given $\imath=\sqrt{-1}$.

Answer: 2

Since $\imath=\sqrt{-1}$, we have $\imath^2=-1$, $\imath^3=\imath^2\times\imath=-\imath$, and $\imath^4=\imath^2\times\imath^2=(-1)^2=1$. So $\imath^{2008}-\imath^{2006}=(\imath^4)^{502}-(\imath^4)^{501}\times\imath^2=(1)^{502}-(1)^{501}\times(-1)=1-(-1)=2$.

46. Find two positive whole numbers a and b, both greater than 1, so that $ab-1$ is the product of two consecutive positive integers and $a+b$ is as small as possible. Express your answer as the ordered pair (a,b) where $a<b$.

Answer: (3,7)

ab is one more than the product of two consecutive positive integers. The first few possibilities are $2+1=3$, $6+1=7$, $12+1=13$, $20+1=21$, $30+1=31$. The first three are prime numbers, and thus have no solutions (a,b) with both a and b greater than 1. Since $21=3\times 7$, $(3,7)$ is a possibility; all we must show is that 10 is the minimum possible sum. Any product of two consecutive integers greater than 20 is also greater than 25. Since the smallest possible sum of a factor pair of a certain number n (a factor pair of n is a pair of integers a and $n\div a$) is $2\sqrt{n}$ (why?), any other sum $a+b$ will be greater than $2\sqrt{25}=10$, and our answer is $(3,7)$.

47. The incircle of $\triangle ABC$ touches $\overline{BC}$ at D, $\overline{AC}$ at E, and $\overline{AB}$ at F. Given $\overline{AF}=3$, $\overline{BD}=4$, and $\overline{CE}=6$, find the perimeter of $\triangle ABC$.

Answer: 26

Since tangents to a circle from the same point are equal, we have $\overline{AF}=\overline{AE}=3$, $\overline{BD}=\overline{BF}=4$, and $\overline{CE}=\overline{CD}=6$. The perimeter of the triangle is $\overline{AB}+\overline{AC}+\overline{BC}=\overline{AF}+\overline{AE}+\overline{BD}+\overline{BF}+\overline{CE}+\overline{CD}=2\times 3+2\times 4+2\times 6=6+8+12=26$.

48. Four farmers find 20 cows in a field and decide to split the cows between them (cows may not be split in half). Farmer A insists on having at least 1 more cow than farmer B, at least 2 more cows than farmer C, and at least 3 more cows than farmer D. What is the minimum number of cows Farmer A can take?

Answer: 7

If farmer A claims only 6 or less cows, the total number of cows claimed is at most $6+5+4+3=18<20$. So the answer must be at least 7, which is possible when Farmer A takes 7 cows, Farmer B takes 6 cows, Farmer C takes 5 cows, and Farmer D takes 2 cows.

49. When Joyce goes out to buy clothes, she buys three shirts, each worth \$12, and a bathing suit worth \$24. If she leaves the store with $\frac{1}{3}$ of the money that she entered the store with, then how many dollars did Joyce enter the store with?

Answer: 90

Joyce spent a total of $3\times \$12+\$24=\$36+\$24=\$60$ in the store, and has half of this amount, or \$30, left. Thus, she entered the store with $3\times \$30=\$90$.

50. Find the sum of the first 16 terms of the arithmetic sequence 3, 7, 11, ...

Answer: 528

The n^{th} term is given by the formula $4n-1$. Thus, terms a and $17-a$ sum to $4a-1+4(17-a)-1=4a-1+68-4a-1=68-2=66$. There are $16\div 2=8$ pairs of terms, so the sum of the first 16 terms of the sequence is $66\times 8=528$.