6th Grade Competition Solutions

Bergen County Academies Math Competition
19 October 2008

1. Before taking the AIME, a student notices that he has two can of soda, three bags of chips, and six bars of chocolate on his desk. If each can of soda has 140 calories, each bag of chips has 120 calories, and each bar of chocolate has 60 calories, and the student devours everything on his desk during the test, then how many calories did the student consume?
Answer: 1000
$2 \cdot 140 + 3 \cdot 120 + 6 \cdot 60 = 280 + 360 + 360 = 1000$.

2. In order to walk completely around his square backyard, Mike must walk 84 yards. How long is one side of his field in yards?
Answer: 21
The perimeter of a square with side length s is $4s$. Setting $4s = 84$, we find $s = \frac{84}{4} = 21$.

3. If $a\#b = 2a - b + 117$, then what is $(3\#4) - (5\#2)$?
Answer: –6
$(3\#4) - (5\#2) = (2 \cdot 3 - 4 + 117) - (2 \cdot 5 - 2 + 117 = 6 - 4 + 117 - 10 + 2 - 117 = 2 - 8 = -6$.

4. An airplane ascends from sea level to 30,000 feet above sea level at 500 feet per minute. How many minutes does it take the airplane to reach its target height?
Answer: 60
It takes 2 minutes for the plane to ascent 1000 feet. Thus, it will take $30 \cdot 2 = 60$ minutes for the plain to ascend 30,000 feet.

5. The sum of five consecutive terms of an arithmetic sequence is 30. If the common difference is 6, then find the difference between the largest and smallest of the five terms.
Answer: 24
If the first of the five terms is a, then the next for terms are $a+6$, $a+12$, $a+18$, $a+24$, and our answer is $a+24-a = 24$. Although the sum of the five terms is irrelevant, it allows us to identify the sequence as $-6, 0, 6, 12, 18$

6. One Friday night, Mr. Holbrook had 219 Tootsie Rolls in his candy cabinet. (Not many people at math team like to eat Tootsie Rolls.) The next morning, he gets a shipment of 87 Tootsie Rolls. How many Tootsie Rolls does Mr. Holbrook have now?
Answer: 306
$219 + 87 = 306$.

7. Find the sum of the first 8 terms in the pattern $26, 23, 20, 17, \ldots$
Answer: 124
The n^{th} term of the arithmetic sequence is given by $29 - 3n$. Thus, the sum of term k and term $9 - k$ is $(29 - 3k) + (29 - 3(9 - k)) = 29 - 3k + 29 - 27 + 3k = 29 + 2 = 31$. There are $\frac{8}{2} = 4$ pairs of terms, so the sum of the first 8 terms of the sequence is $4 \cdot 31 = 124$.

8. $6 + 66 + 666 + 6666 + 66666 = ?$
Answer: 74070
By the distributive law, $6 + 66 + 666 + 6666 + 66666 = 6(1 + 11 + 111 + 1111 + 11111) = 6 \cdot 12345 = 74070$.

9. The formula for converting Fahrenheit temperatures to Celsius temperatures is $C = \frac{5}{9}(F - 32)$. Convert 50^o Fahrenheit to degrees Celsius.
Answer: 10

Using the formula, our answer is $\frac{5}{9}(50-32) = \frac{5}{9} \cdot 18 = 10$.

10. Find the largest prime factor of 111.
Answer: 37
111 is clearly divisible by 3. Factoring this out, we have $111 = 3 \cdot 37$. Since 37 is not divisible by any of 2, 3, and 5, it is prime. (To check if a number is prime, we need only verify that it is not divisible by any prime number less than or equal to its square root.) So the answer is 37.

11. $9 \cdot (47+53) = 3 \cdot (7+?)$
Answer: 293
Let $x = 7+?$. Then $9 \cdot 100 = 3x \Rightarrow x = 300 \Rightarrow 7+? = 300 \Rightarrow ? = 293$.

12. $9 \cdot (1001+999) = .9 \cdot (10001+ ?)$.
Answer: 9999
Dividing by .9 on both sides, we find that $10 \cdot 2000 = 10001+? \Rightarrow 20000 = 10001+? \Rightarrow ? = 9999$.

13. 15% of a number is equal to 9. Find the number.
Answer: 60
If our number is x, then $.15x = 9 \Rightarrow \frac{3}{20}x = 9 \Rightarrow x = 20 \cdot \frac{9}{3} \Rightarrow x = 60$.

14. Alex Anderson is typing up notes for his geometry class. If it takes him a half hour to make a diagram on the computer and fifteen minutes to type each solution, then how many hours will it take him to type up four solutions and two diagrams?
Answer: 2
$4 \cdot .25 + 2 \cdot .5 = 1 + 1 = 2$.

15. Find the area of a circle with circumference 24π.
Answer: 144π
The circumference of a circle with radius r is $2\pi r$. Setting $2\pi r = 24\pi$, we find $r = 12$. Now, the area of a circle with radius r is $\pi r^2 = \pi \cdot 12^2 = 144\pi$.

16. Ben drives at 40 miles per hour towards North Dakota, 2000 miles away. Halfway there, he decides that he doesn't want to go to North Dakota, and also that he should travel faster. Thus, he turns back home, and drives back at an average speed of 50 miles per hour. How long, in hours, did Ben's trip take?
Answer: 45
Ben went a distance of 1000 miles both ways. The first 1000 miles took him $\frac{1000}{40} = 25$ hours to complete while the second 1000 miles took him $\frac{1000}{50} = 20$ hours, and our answer is $20 + 25 = 45$ hours.

17. What is the prime factorization of 6534?
Answer: $2 \cdot 3^3 \cdot 11^2$
Clearly, 6534 is divisible by 2, and we have $6534 = 2 \cdot 3267$. Also, 3267 is divisible by 9, so we have $6534 = 2 \cdot 3^2 \cdot 363$. 363 is divisible by 3, so $6534 = 2 \cdot 3^3 \cdot 121$. Since $121 = 11^2$, $6534 = 2 \cdot 3^3 \cdot 11^2$, and we are done.

18. If n is a positive integer, then how many values of $\imath^n$ are distinct? $\imath = \sqrt{-1}$.
Answer: 4
$\imath^1 = \imath$, $\imath^2 = -1$, $\imath^3 = \imath^2 \cdot \imath = -\imath$, and $\imath^4 = (-1)^2 = 1$. $\imath^k$, where $k > 4$, is the same as $\imath^4 \cdot \imath^{k-4} = 1 \cdot \imath^{k-4} = \imath^{k-4}$, so there are no more distinct values.

19. What is the smallest positive number that must be added to 83 to get a number that ends in two 0's?
Answer: 17
The closest number to 83 that ends in two zeroes is 100, and $100 - 83 = 17$.

20. During the first quarter of a two-hour flight, Jeff can see the ground the entire time. However, exactly one-quarter of the way into the flight, clouds begin to block Jeff's view of the ground. If the cloud cover persists until the end of the flight, how many seconds of the flight could Jeff not see the ground?

Answer: 5400

The two-hour flight lasted $2 \cdot 60 = 120$ minutes, which is the same as $120 \cdot 60 = 7200$ seconds. Jeff could not see the ground for $\frac{3}{4}$ of this time, or $3 \cdot \frac{7200}{4} = 3 \cdot 1800 = 5400$ seconds.

21. Find the last digit of the number $2008^2 + 2^{2008}$.

Answer: 0

Since 8^2 ends in a 4, 2008^2 will also end in a 4. For the 2^{2008} part, we use a well known fact: that the units digit of a power of an integer cycles through only some possible values. Since $2^1 = 2$, $2^2 = 4$, $2^3 = 8$, and $2^4 = 16$, we see that the pattern for 2 is 2, 4, 8, 6 (and it repeats after the 6 since $2 \cdot 6$ ends in a 2). Since 2008 leaves a remainder of 0 when divided by 4, 2^{2008} will end in a 6, and $2008^2 + 2^{2008}$ will end in the same number that $4 + 6 = 10$ ends in, which is 0.

22. Compute 8635×2121.

Answer: 18,314,835

By straight calculation, $8635 \cdot 2121 = 18,314,835$.

23. Each day David does not shower, the number of bacteria on his body doubles. Every day he does shower, $\frac{15}{16}$ of the bacteria on his body are killed. David comes to math camp with 6 bacteria on his body and doesn't shower for a week. The next day, he showers and then leaves math camp and arrives home. How may bacteria are on his body when he gets home?

Answer: 48

After the week of not showering, David has $6 \cdot 2^7$ bacteria on his body. Showering reduces this number by a factor of 16, so after showering, David has $6 \cdot 2^{7-4} = 6 \cdot 2^3 = 6 \cdot 8 = 48$ bacteria on his body. This is the same number of bacteria with which he arrives home.

24. Students at math team consume bags of chips at a rate that is directly proportional to the time that they have been at math team that day. If a math team member consumes three bags of chips per hour after being at math team for one hour, then what is the rate of consumption of chips, in bags per hour, of a member who has been at math team for three hours?

Answer: 9

If r is the hourly rate at which chips are consumed and t is the time the math team member has been at math team, then $\frac{r}{t} = k$, for k a constant. By the given information, we know that $k = \frac{3}{1} = 3$. Setting $\frac{r}{3} = 3$, we find that this math team member will consume $3 \cdot 3 = 9$ bags of chips per hour.

25. During his first week in America, Tolga goes to Six Flags and spends \$40 on games. If all he wins is a basketball, and he sells the basketball to his friend John for \$5 at Six Flags, then how many more dollars did Tolga enter Six Flags with than he left it?

Answer: 35

Tolga spends \$40 at Six Flags, but makes \$5, so he leaves with \$40 – \$5 = \$35 more than he came with.

26. What is the volume of a sphere with radius $\dfrac{12}{\sqrt[3]{\pi}}$?

Answer: 2304

The volume of a sphere with radius r is $\frac{4}{3}\pi r^3$. In this case, that is $\dfrac{4}{3}\pi\left(\dfrac{12}{\sqrt[3]{\pi}}\right)^3 = \dfrac{4}{3} \cdot 12^3 = 16 \cdot 12^2 = 2304$.

27. John draws a card from a standard deck of 52 cards, replaces it, and then chooses another card. What are the chances that he chooses a spade the first time and a four the second time?

Answer: $\frac{1}{52}$

The chances of choosing a spade the first time are $\frac{1}{4}$. The chances of choosing a four the second time are $\frac{1}{13}$. The chances of both of these happening in the order given are $\frac{1}{4} \cdot \frac{1}{13} = \frac{1}{52}$

28. A palindrome is a number that is read the same way forwards as it is backwards. For example, 70107 is a palindrome, but 502 is not. Find the sum of all two-digit palindromes.
Answer: 495
Each two-digit palindrome is a multiple of 11. The sum we want is $11+22+33+\ldots+99 = 11(1+2+\ldots+9) = 11\cdot 45 = 495$.

29. Fully simplify $3\sqrt{27\sqrt{9}}$.
Answer: 27
$3\sqrt{27\sqrt{9}} = 3\sqrt{27\cdot 3} = 3\sqrt{81} = 3\cdot 9 = 27$.

30. Compute the length of a diagonal of rectangle $ABCD$ given $\overline{AB} = 9$ and that the area of $ABCD$ is 81.
Answer: $9\sqrt{2}$
The other side of this rectangle must be $\frac{81}{9} = 9$, making this a square. By the Pythagorean Theorem, the length of a diagonal must be $\sqrt{9^2+9^2} = \sqrt{81+81} = 9\sqrt{1+1} = 9\sqrt{2}$.

31. One day, Mark was mad, and dug a 7-foot deep hole in his backyard. However, since he didn't toss the dirt far enough away from the hole, 8 inches of dirt fell back into the hole. How deep in inches is the hole now?
Answer: 76
The hole was originally $7\cdot 12 = 84$ inches deep. When 8 inches of dirt fell back into it, the hole became $84-8 = 76$ inches deep.

32. At what point do the lines $2x+3y = 5$ and $x+y = 2$ intersect? Give your answer in the form (x, y), including the parentheses and the comma.
Answer: (1,1)
Subtracting twice the second equation from the first, we obtain $y = 1$. Plugging this in to the second equation, we find $x+1 = 2 \Rightarrow x = 1$, and $(x, y) = (1, 1)$.

33. Given points $A(3,4)$ and $B(5,12)$, find the distance $\overline{AB}$.
Answer: $2\sqrt{17}$
The distance from a point with coordinates (x_1, y_1) to a point with coordinates (x_2, y_2) is $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$. In this case, that is $\sqrt{(5-3)^2+(12-4)^2} = \sqrt{2^2+8^2} = \sqrt{4+64} = 2\sqrt{17}$.

34. Find the number of two-digit numbers n such that the sum of n and the number obtained by switching the digits of n is divisible by 11.
Answer: 90
In fact, every two-digit number has this property. Consider $x = 10a+b$, for a and b digits. Then the sum specified is $(10a+b)+(10b+a) = 11a+11b$, which is clearly divisible by 11. Finally, there are 90 two-digit numbers, and our answer is 90.

35. Tom has an 89 average on his first two tests of the year. If the maximum possible grade on any test is a 100, find the lowest possible grade he can get on his next test in order to be able to get a 93 average at the end of the year (there are 4 tests in Tom's class this year).
Answer: 94
If Tom wants a 93 average by the end of the year, he needs to earn $4\cdot 93 = 372$ total points. The minimum possible score on the third test occurs with the maximum possible score on the fourth test. So the minimum possible score on the third test is $372-(2\cdot 89+100) = 372-(178+100) = 372-278 = 94$.

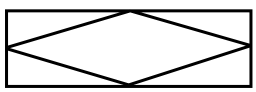
36. Jenny buys a twelve pack of soda cans at the gas station for \$2.40. How many cents did she pay for each can?
Answer: 20

Jenny pays $\dfrac{240}{12} = 20$ cents for each can of soda.

37. The next day (refer to problem 36), Jenny goes back to the gas station and buys another twelve pack of soda, but this time, it costs \$3.60. How many more cents did she pay for a can of soda this time than she did in problem 36?
Answer: 10
Jenny pays $\$3.60 - \$2.40 = \$1.20$ more for the second twelve pack, which is the same as $\dfrac{120}{12} = 10$ cents more per can.

38. Ben is trying to throw darts at a dartboard. Each throw he makes, the dart ends up at most one foot away from some part of the dartboard (sometimes, he hits the dartboard). Also, any point in this area has an equal chance of being hit. If the dartboard is a circle of radius two feet, then what is the probability that Ben hits the dartboard?
Answer: $\frac{4}{9}$
Each throw, Ben hits a point in a circle of radius 3, and the dartboard is a circle of radius 2. The chances of Ben hitting the dartboard are the ratio of these two areas, or $\dfrac{\pi 2^2}{\pi 3^2} = \dfrac{4\pi}{9\pi} = \dfrac{4}{9}$.

39. Find the area of a rhombus with diagonals of lengths 2 and 18.
Answer: 18
Consider a rectangle around the rhombus with sides parallel to the diagonals of the rhombus, as shown. The area of the rhombus is $\frac{1}{2}$ that of the rectangle, or $\dfrac{1}{2} \cdot 2 \cdot 18 = 18$.



40. How many 2×3 tiles does it take to cover a 6×6 square?
Answer: 6
The area of the square is $6^2 = 36$. Since the area of each tile is $2 \cdot 3 = 6$, it will take $\dfrac{36}{6} = 6$ tiles to cover the square.

41. If the three solutions to $x^3 - 12x^2 + 27x = 0$ are a, b, and c and $a \le b \le c$, compute $a + c^b$.
Answer: 729
Clearly, x factors, so we have the equation $x(x^2 - 12x + 27) = 0$. The quadratic also factors, so we are left with $x(x-3)(x-9) = 0$. The solutions to the equation are $(a, b, c) = (0, 3, 9)$, and $a + c^b = 0 + 9^3 = 729$.

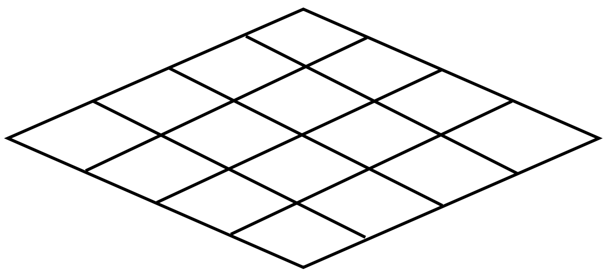
42. A coin is flipped five times. What is the probability of heads coming up exactly twice?
Answer: $\frac{5}{16}$
The 5 coin flips can occur in $\dbinom{5}{2} = \dfrac{5!}{2! \cdot 3!} = 10$ ways. The probability of flipping any sequence of 5 flips is $\left(\dfrac{1}{2}\right)^5 = \dfrac{1}{32}$, so our answer is $10 \cdot \dfrac{1}{32} = \dfrac{5}{16}$.

43. Two positive numbers are relatively prime if they share no factors other than 1. How many numbers less than or equal to 21 are relatively prime to 21?
Answer: 12
Any number that is not a multiple of 3 or 7 is relatively prime to 21. There are $\frac{21}{3} = 7$ multiples of 3 less than or equal to 21 and $\frac{21}{7} = 3$ multiples of 7 less than or equal to 21. But we have counted 21 twice, so there are $21 - 7 - 3 + 1 = 12$ numbers less than or equal to 21 that are relatively prime to 21.

44. A sphere fits exactly inside of a cube – it just touches all six faces of the cube. If the volume of the cube is 27, then what is the radius of the sphere?
Answer: $\frac{3}{2}$

The radius of the sphere will be one-half the length of a side of the cube. The length of a side of the cube is $\sqrt[3]{27} = 3$, and so the radius of the sphere is $\frac{3}{2}$.

45. What is the least number of colors needed to color each of the parallelograms below such that no two parallelograms that share an edge are colored the same color?
Answer: 2
The coloring cannot be done only one color, but it can be done with 2 by coloring the drawing like a checkerboard.

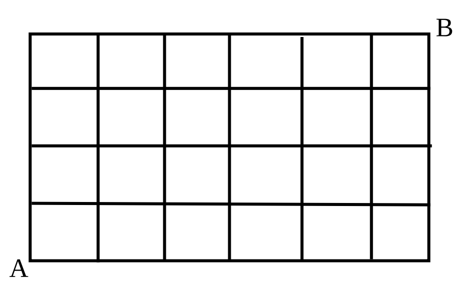


46. Tolga is throwing a basketball at the wall. Each time he throws it, the strength of the wall is reduced to $\frac{9}{10}$ of its strength before that throw. If the wall was at full strength before Tolga started throwing the ball at the wall and if the wall breaks when its strength is reduced to below 50% of its original strength, then the wall will break after the n^{th} throw. Find n.
Answer: 7
We need to find the smallest n such that $100 \cdot .9^n < 50$. If $n = 6$, this expression is 53.1, but when $n = 7$, the expression is equal to 47.8.

47. A golf coach distributes 54 tees and 30 golf balls to a team of n people so that each of the n players gets the same number of tees and each player gets the same number of balls (although the number of tees each player receives may be different from the number of balls they receive). If there are no tees or balls left over, then what is the largest possible value of n?
Answer: 6
The question asks for the greatest common factor of 54 and 30. Since $54 = 2 \cdot 3^3$ and $30 = 2 \cdot 3 \cdot 5$, a factor of both of these must have at most one factor of 2 and one factor of 3, but no more prime factors. The greatest prime factor will have both of these, and that number is $2 \cdot 3 = 6$.

48. How many composite numbers are less than 30?
Answer: 18
There are 10 prime numbers less than 30: 2, 3, 5, 7, 11, 13, 17, 19, 23, and 29. Since 1 is not composite either, there are $29 - 10 - 1 = 18$ composite numbers less than 30.

49. In how many ways can Lena get from point A to point B if Lena can only move up and right?
Answer: 210



Lena must make 10 moves – 4 up and 6 right. The sequence of moves can be made in $\binom{10}{4} = \frac{10!}{4! \cdot 6!} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} = 210$ different ways.

50. What is the distance from the origin to the line $8x + 6y = 72$?
Answer: $\frac{36}{5}$
$8x + 6y = 72 \Rightarrow 4x + 3y = 36 \Rightarrow 3y = 36 - 4x \Rightarrow y = 12 - \frac{4}{3}x$. This graph, along with the coordinate axes, bound a right triangle with right angle at the origin and legs on the axes. Since the intercepts of the graph are 12 and $\frac{12}{\frac{4}{3}} = \frac{12 \cdot 3}{4} = 9$, the sides of the triangle have lengths 9, 12, and $\sqrt{9^2 + 12^2} = \sqrt{81 + 144} = \sqrt{225} = 15$. Now we must find the length of the altitude to the hypotenuse. The area of the triangle is $\frac{1}{2} \cdot 12 \cdot 9 = 6 \cdot 9 = 54$. Using the hypotenuse as the base, we find our desired distance h satisfies $\frac{1}{2} \cdot 15h = 54 \Rightarrow h = \frac{108}{15} \Rightarrow h = \frac{36}{5}$.