7th Annual Bergen County Academies Math Competition

Fifth Grade

Sunday, 18 October 2009

1. If $100 \cdot (1.9 + 9.1) = 10 \cdot (28 + x)$, what is x?

   Solution: Dividing both sides by 10 and simplifying the left side, one gets $110 = 28 + x$. Solving for x by subtracting 28 from both sides results in $\boxed{x = 82}$.

2. Josh is thinking of two numbers from the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$. Their sum is 9. How many possible values are there for the smaller of the two numbers?

   Solution: Josh could be thinking of 1 and 8 (first pair), 2 and 7 (second pair), 3 and 6 (third pair), or 4 and 5 (fourth pair); this shows that there are exactly four possible pairs, and thus $\boxed{4}$ possible values for the smaller of the two numbers.

3. $(2 \cdot 2009) + (4 \cdot 2009) + (6 \cdot 2009) = ? \cdot 2009$

   Solution: Replacing the unknown quantity with the variable x and factoring out 2009 from the expression on the left, we get

   $$2009 \cdot (2 + 4 + 6) = x \cdot 2009.$$

   Dividing both sides by 2009 gives $x = 2 + 4 + 6 = \boxed{12}$.

4. Complete the following sequence: 1, 2, 4, 4, 9, 8, 16, 16, 25, 32, ?

   Solution: Look at alternating terms. 1, 4, 9, 16, 25, ... are squares. 2, 4, 8, 16, 32, ... are powers of 2. The next term continues the first sequence, so it is $\boxed{36}$.

5. Evaluate $1 - 2 - 3 + 4 + 5 - 6 - 7 + 8 + 9 - ... + 2009$.

   Solution: We can rewrite this as $1 + (-2 - 3) + (4 + 5) + (-6 - 7) + (8 + 9) + \cdots + 2009 = 1 + (-5) + 9 + (-13) + 17 + \cdots + 4017$. This, in turn, may be written as $1 + ((-5) + 9) + ((-13) + 17) + \cdots + ((-4013) + 4017) = 1 + (9 - 5) + (17 - 13) + \cdots + (4017 - 4013) = 1 + 4 + 4 + \cdots + 4$. In the original series, every four terms (not including the first term) summed to 4. There were 2009 terms in the series, and thus there are $(2009 - 1)/4 = 2008/4 = 502$ 4's in the new series. Thus, the final sum is $502 \cdot 4 + 1 = 2008 + 1 = \boxed{2009}$.

6. Jenny is rearranging the salt and pepper shakers in a restaurant. There are four salt shakers and five pepper shakers. All the salt shakers are indistinguishable and all the pepper shakers are indistinguishable. How many ways can she arrange them in a line such that the salt and pepper shakers alternate?

   Solution: We have two possible cases for the first shaker: it can be a salt shaker or a pepper shaker. If it is a salt shaker, then the next one must be a pepper shaker, and we can continue this and get the arrangement SPSPSPSP (the letter S represents a salt shaker, and the letter P represents a pepper shaker), where the next shaker must be salt. However, we have no more salt shakers and still have a pepper shaker remaining, so this case cannot happen. If we begin with a pepper shaker, then by the same reasoning we will eventually obtain the single forced arrangement PSPSPSPSP, which is legal, so the number of arrangements is $\boxed{1}$.

7. Find the value of x given the following equations:

$$3 \cdot x - 4 \cdot y = 2$$

$$5 \cdot y = 2 - 4$$

Solution: To solve for x, we must first solve for y. Since y is the only variable in the second equation, we can isolate it by dividing by 5 on both sides of this equation. This gives $y = \frac{2-4}{5} = \frac{-2}{5}$. Then, in the first equation, we isolate $3x$ by adding $4y$ to both sides to get

$$3x - 4y + 4y = 2 + 4y$$

$$3x = 2 + 4y = 2 + 4\left(\frac{-2}{5}\right) = 2 - \frac{8}{5} = \frac{10}{5} - \frac{8}{5} = \frac{10-8}{5} = \frac{2}{5}$$

so $3x = \frac{2}{5}$. Dividing by 3 on both sides gives $\boxed{x = \frac{2}{15}}$.

OR

Solution: Multiply the top equation by 5, and the bottom equation by 4, to get

$$15 \cdot x - 20 \cdot y = 10$$

$$20 \cdot y = -8$$

Then add the two equations to get $15x = 2$, or $\boxed{x = \frac{2}{15}}$.

8. On Steve's farm, there are chickens (which have one head and two legs), and headdie-horses (which have three heads and five legs). Richard decided to count the heads and legs on the farm. He counted forty-three heads and seventy-nine legs. How many chickens were there?

   Solution: Let C be the number of chickens and H be the number of horses. The problem gives us the equations $C + 3 \cdot H = 43$ and $2 \cdot C + 5 \cdot H = 79$. To solve for C, we multiply the first equation by 5 and the second equation by 3 to get $5 \cdot C + 15 \cdot H = 215$ and $6 \cdot C + 15 \cdot H = 237$. Subtracting the first equation from the second, we get $C = \boxed{\text{22 chickens}}$.

9. Find the twentieth positive odd number.

   Solution: The first positive odd number is 1. After that comes 3, 5, 7, 9, ... , with the twentieth being $\boxed{39}$.

10. A rectangular prism has two faces with area 6, two faces with area 18, and two faces with area 300. Find its volume.

    Solution: We label the different edges a, b, and c. Either $a \cdot b = 6$, $b \cdot c = 18$ and $a \cdot c = 300$, or $a \cdot b = 6$, $a \cdot c = 18$ and $b \cdot c = 300$. If we multiply these all together in either case, we get that $(a \cdot b \cdot c)^2 = 18 \cdot 300 \cdot 6 = (18) \cdot (3 \cdot 100) \cdot (6) = (18) \cdot (3 \cdot 6) \cdot (100) = (18 \cdot 18) \cdot (10 \cdot 10) = (18 \cdot 10)^2 = 180^2$. Thus, $a \cdot b \cdot c$, the volume of the prism, equals $\boxed{180}$.

11. A car and truck are traveling toward each other. The car travels at 20 miles per hour and the truck travels at 30 miles per hour. If they were originally 150 miles apart, after how many hours do they meet?

    Solution: Let t be the time, in hours, that it takes the two vehicles to meet. In this time, the car travels a distance of $20t$ miles and the truck travels a distance of $30t$. So the total distance they travel is $20t + 30t = (20 + 30)t = 50t$. This has to equal 150 miles, so $50t = 150$; dividing both sides by 50 gives $t = \boxed{\text{3 hours}}$.

12. How many factors does 1001 have?

    Solution: The prime factorization of 1001 is $7 \cdot 11 \cdot 13 = 7^1 \cdot 11^1 \cdot 13^1$. The number of factors can by calculated by finding the product of the power of each prime in the factorization plus 1 (this is true for any number). In this case, $(1+1) \cdot (1+1) \cdot (1+1) = 8$, so 1001 has $\boxed{\text{8 factors}}$.

13. Alex Zhu was asked to find the sum of two nonnegative integers. Instead, he found their product. Luckily, his answer was the same. Find all possible values of the sum of those two integers.

    Solution: Let m and n be the two nonnegative integers. Then $m + n = mn$; subtracting n from both sides gives

$$m + n - n = mn - n$$

$$m = (m-1)n$$

and $n = \frac{m}{m-1}$. The only time that $m - 1$ divides m is if $m - 1 = 1$ or $m = 0$. In the first case, $m = 2$ and $n = 2$, so $m + n = 4$. In the second case, $n = 0$, so $m + n = 0$. Thus, the two possible sums are $\boxed{\text{0 and 4}}$.

14. Simplify the following expression: $\dfrac{60}{32} \cdot \dfrac{12}{14} \cdot \dfrac{28}{45}$.

    Solution: Cancel out 28 with 14 to get $\frac{60}{32} \cdot 12 \cdot \frac{2}{45}$. Cancel out 60 with 45 to get $\frac{4}{32} \cdot 12 \cdot \frac{2}{3}$. Cancel out 12 with 3 to get $\frac{4}{32} \cdot 4 \cdot 2$. Multiply to get a complete numerator of 32. Simply $\frac{32}{32}$ to $\boxed{1}$.

15. Sherry's rectangular garden measures 5 feet wide by 4 feet long. If she wants to plant potatoes on the perimeter of the garden such that each potato is at least a foot away from any other potato, what is the maximum number of potatoes she can plant?

    Solution: On the top edge of the garden, you can plant 6 potatos if they are spaced 1 foot from each other. The same goes for the bottom edge. Then, on the left side, you can plant 3 additional potatoes (you've already planted in the top-left and bottom-left corners, so those spaces are already filled), and the same on the right. Thus, you can plant $2 \cdot 6 + 2 \cdot 3 = 12 + 6 = \boxed{\text{18 potatoes}}$.

16. What is the sum of the first 20 positive multiples of 3?

    Solution: The first positive multiple of 3 is 3, and the 20th is 60. So consider the sum $3 + 6 + 9 + 12 + \cdots + 60$. Divide out by 3 to get $3 \cdot (1 + 2 + 3 + 4 + \cdots + 20) = 3 \cdot ((1 + 20) + (2 + 19) + (3 + 18) + (4 + 17) + \cdots + (10 + 11)) = 3 \cdot (21 + 21 + 21 + 21 + \cdots + 21)$. Since we were originally summing 20 terms, and then paired them up, this is a sum of $\frac{20}{2} = 10$ twenty-ones. The sum inside the parenthesis is $21 \cdot 10 = 210$. Multiply by three to get the complete solution: $\boxed{630}$.

17. If Gary drives to Peter's house at 50 miles per hour, it will take him 2 hours less than if he bikes to Peter's house at 10 miles per hour. How far away does Gary live from Peter's house, in miles?

    Solution: Let t be the time, in hours, it takes for Gary to <u>bike</u> to Peter's house. Then $t - 2$ is the time, in hours, it takes for Gary to <u>drive</u> to Peter's house. If d is the distance, in miles, between their houses, then we have the system of equations

$$d = 10 \cdot t$$

$$d = 50 \cdot (t - 2)$$

By noticing that this means $10t = 50(t-2)$, we can solve for t. $50(t-2) = 50t - 100 = 10t$; subtracting $10t$ from both sides and adding 100 to both sides gives $40t = 100$, and dividing both sides by 40 gives $t = \frac{5}{2}$. Plugging this into the first equation, we get $d = 10 \cdot \left(\frac{5}{2}\right) = \boxed{\text{25 miles}}$.

18. Find the number of integer solutions to the equation $x^2 + y^2 = 103$.

Solution: There are very few possibilities for x^2. x^2 equals one of the following: 0, 1, 4, 9, 16, 25, 36, 49, 64, 81 or 100. y^2 then equals 103, 102, 99, 94, 87, 78, 67, 54, 39, 22 or 3, respectively. None of these values yield integer values for y, so the number of solutions is $\boxed{0}$.

OR

Solution: Squares are either 0 or 1 mod 4. 103 is 3 mod 4, thus the number of solutions is clearly $\boxed{0}$.

19. Julia is playing darts. On each turn, she can score 5 or 7 points. What is the maximum number of points she cannot get?

Solution: In particular, Julia can get 0, 7, 14, 21, and 28 points. If we order these by their remainders when divided by 5, we get 0, 21, 7, 28, 14. We can add multiples of 5 to 0 in order to get any other number with a remainder of 0, we can add multiples of 5 to 21 in order to get any other number greater than 21 with a remainer of 1, etc. However, while we can add 5's, we cannot subtract 5's; therefore, there are no numbers less than 21 with a remainder of 1 that are obtainable, no numbers less than 7 with a remainder of 2 that are obtainable, etc. Therefore, the largest unobtainable number is $28 - 5 = \boxed{23}$.

OR

Solution: There is an identity that says that the largest impossible score will be $7 \times 5 - 7 - 5 = \boxed{23}$.

20. Find the number of subsets S, including the empty set, of $\{1, 2, 3, ..., 10\}$ such that if $x \neq 10$ and x is in S, then $x + 1$ is also in S.

Solution: Let z be the smallest element of a subset T. Clearly, $z + 1$ is an element of T, as is $z + 2$, $z + 3$, $z + 4$, etc., up to 10. This means that each smallest element corresponds to a unique set (ex. 7 corresponds to $\{7, 8, 9, 10\}$ and 4 corresponds to $\{4, 5, 6, 7, 8, 9, 10\}$). This means that there are 10 non-empty subsets and 1 empty subset, leading to a total of $\boxed{\text{11 subsets}}$.

21. Alex Zhu has suggested that the BCA math competition have 50 normal problems worth 1 point each, 5 hard problems worth 2 points each, and 10 very hard problems worth 3 points each. If this were so, find the maximum possible score that a student could get.

Solution: 50 hard problems worth 1 point each can yield up to $50 \cdot 1 = 50$ points. 5 hard problems worth 2 points each can give $5 \cdot 2 = 10$ points. 10 really hard problems worth 3 points each are worth a total of $10 \cdot 3 = 30$ points. Adding these up, we find that the highest possible score is $50 + 10 + 30 = \boxed{\text{90 points}}$.

22. If bagels are only sold in bags of 17, what is the least number of bags I must buy to get 103 bagels?

Solution: If n is the number of bags needed, then $17 \cdot n \geq 103$, or $n \geq \frac{103}{17}$. Thus, the problem is asking us to find the least integer larger than $\frac{103}{17}$. $\frac{103}{17} > 6$ because $6 \cdot 17 = 102$, so the next integer is $\boxed{7}$, which is indeed greater than $\frac{103}{17}$.

23. What is 9950.0599 rounded to the nearest hundred?

Solution: The hundreds digit is 9, and the digit immediately to its right (the tens digit) is 5. Since the tens digit is ≥ 5, we have to round up. So we round the 9 to 10, and carry the 1. The 9 in the thousands place then becomes a 10, and the final answer is $\boxed{10,000}$.

24. Ben buys a sharpie and two rulers. The total cost is \$2.25. If a sharpie costs \$0.50 more than a ruler, how much would it cost to buy four sharpies and three rulers?

Solution: Let s be the cost of one sharpie, and let r be the cost of one ruler. Then we have the equations

$$s + 2r = 2.25$$
$$s = r + 0.50$$

Subtracting the second equation from the first, we have $2r = 2.25 - r - 0.50$, or $3r = 1.75$. This means that $r = \frac{1.75}{3} = \frac{175}{300} = \frac{7}{12} = 0.58\overline{3}$. Plugging this value of r into the second equation, we get $s = \frac{7}{12} + 0.50 = \frac{13}{12} = 1.08\overline{3}$. To buy four sharpies and three rulers costs $4s + 3r = 4 \cdot \frac{13}{12} + 3 \cdot \frac{7}{12} =$ $\boxed{\$\frac{73}{12} = \$6\frac{1}{12} = \$6.08\overline{3} = \$6.0833\ldots \approx \$6.08}$.

25. How many integers n are there such that $\frac{120}{n}$ is also an integer?

    Solution: The positive divisors of 120 are 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, and 120. The 16 of these and their 16 negatives all yield integers when dividing 120, so there are $2 \cdot 16 = \boxed{32}$ possible values of n such that $\frac{120}{n}$ is an integer.

26. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. Given that $\pi \approx 3.14159265\ldots$, compute $\lfloor 203\pi \rfloor$.

    Solution: The greatest integer function will take a number like 17.17 and remove the decimal part, giving $\lfloor 17.17 \rfloor = 17$. Therefore, we are only interested in computing the whole number part of 203π.

    $203\pi \approx 203 \cdot 3.14159265\ldots = 203 \cdot (3 + 0.1 + 0.04 + 0.001 + 0.0005 + \cdots) = 609 + 20.3 + 8.12 + 0.203 + 0.1015 + \cdots$. It is unnecessary to compute any more terms, as they will not contribute to the integer part. Adding up what we have so far, we find that $203\pi \approx 637.7245$, the integer part of which is $\boxed{637}$.

27. Alice flips a coin four times. What is the probability that she will get exactly one head?

    Solution: The one heads can either come as the result of the first, second, third, or fourth flip, and thus there are 4 ways to get only one heads. On each of the 4 flips, there are 2 possible outcomes, so the total number of possible outcomes is $2 \cdot 2 \cdot 2 \cdot 2 = 2^4 = 16$. This means that the probability of flipping exactly one heads is $\frac{4}{16} = \boxed{\frac{1}{4}}$.

28. Julia has \$10. Her grandmother returns home from China and gives her \$200. Find the percent increase in Julia's money.

    Solution: The formula for percent increase is $\frac{\text{change}}{\text{initial}} \cdot 100\%$. Plugging in the numbers for this problem, we have percent change $= \frac{200}{10} \cdot 100\% = 20 \cdot 100\% = \boxed{2000\%}$.

29. Define $x\%y = \dfrac{2 \cdot x + y}{x - y}$.What is $(5 \cdot (5\%2))\%10$?

    Solution: Applying the order of operations, one evaluates $5\%2 = \frac{2\cdot5+2}{5-2} = \frac{12}{3} = 4$. We then multiply this by 5 to get 20. Finally, we do $20\%10 = \frac{2\cdot20+10}{20-10} = \frac{50}{10} = \boxed{5}$.

30. Mark and Dan are racing against each other. Dan can run at 1 mile per hour and Mark can run at 15 miles per hour. Dan starts the race 70 miles ahead of Mark. How many hours does it take for Mark to catch up to Dan?

    Solution: Mark is "closing in" on Dan at 14 miles per hour (15 minus 1). To span the 70 mile gap, Mark will need $\frac{70}{14} = \boxed{5 \text{ hours}}$.

31. Find the missing number: $\dfrac{55 \cdot 77}{?} = 5 \cdot 7$

Solution: If we let x be the missing quanitity, we can rewrite the left-hand side of the equation as $\frac{55 \cdot 77}{x} = \frac{5 \cdot 11 \cdot 7 \cdot 11}{x} = 5 \cdot 7 \cdot \frac{11^2}{x}$. We can then divide both sides of the equation by $5 \cdot 7$, getting $\frac{11^2}{x} = 1$; multiplying both sides of the equation by x gives $x = 11^2 = \boxed{121}$.

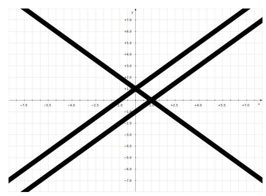
32. A piece of plastic food wrap measures 12 inches by 3000 feet. What is the area in square feet?

Solution: To make the units equal, notice that the food wrap is 1 foot by 3000 feet. The area is base time height, $\boxed{\text{3000 square feet}}$.

33. Find the two points where at least two of lines $y = x + 1$, $x + y = 1$, and $x = y + 1$ intersect.

Solution: First we rewrite the equations in terms of y. These equations are:

$$y = x + 1 \quad (1)$$
$$y = -x + 1 \quad (2)$$
$$y = x - 1 \quad (3)$$



Lines (1) and (3) are parallel because they both have a slope of 1, so they will never intersect with each other. Line (2), on the other hand, will intersect with each line exactly once, so we must solve for those intersections.

First, the intersection between (1) and (2) can be found by equating them. $x + 1 = -x + 1$, so $2x = 0$ and $x = 0$. Plugging this back in, $y = x + 1 = 0 + 1 = 1$, so the point of intersection is $(0, 1)$.

Then, the intersection between (2) and (3) is found by saying $-x + 1 = x - 1$. Then $2x = 2$, and $x = 1$. This leads to finding that $y = x - 1 = 1 - 1 = 0$, so this point of intersection is $(1, 0)$.

Thus, the two points of intersection are $\boxed{(0, 1) \text{ and } (1, 0)}$.

34. Matt, Robert, and Nikhil are dividing a pile of chocolates. Matt takes half of the pile and then takes three more chocolates. Robert then takes half of the pile and then takes three more chocolates. Nikhil then takes half of the pile and then takes three more chocolates. After that, Jordan passes by to collect the last remaining chocolate. How many chocolates were there in the initial pile?

Solution: Let C be the number of chocolates initially. We have that $((C/2-3)/2-3)/2-3 = 1$. Solving backwards, we get $(C/2 - 3)/2 - 3 = 8$. Further we have $C/2 - 3 = 22$. Finally, $C = \boxed{\text{50 chocolates}}$.

35. What is the product of the all the integers from -10 to 10 inclusive?

Solution: This set of numbers includes 0. Since the product of any number and 0 is itself 0, the final product is also $\boxed{0}$.

36. Paul eats 2100 calories of food per day. He wants no more than 15% of his calories to come from fat. If each gram of fat provides nine calories, what is the maximum number of grams of fat that Paul can eat in one day?

Solution: 15% of Paul's diet equals $2100 \cdot 0.15 = 315$ calories. If we divide by 9 calories per gram of fat, we find that Paul can eat a maximum of $\frac{315}{9} = \boxed{\text{35 grams of fat per day}}$.

37. What is $2\dfrac{2}{3}$ subtracted from its reciprocal?

Solution: First, $2\frac{2}{3}$ must be converted to an improper fraction; this would be $\frac{2\cdot3+2}{3} = \frac{8}{3}$. The reciprocal of $\frac{8}{3}$ is $\frac{3}{8}$, and the desired difference is $\frac{3}{8} - \frac{8}{3} = \frac{9}{24} - \frac{64}{24} = \frac{9-64}{24} = \boxed{-\dfrac{55}{24}}$. In mixed fraction form, this would be $\boxed{-2\dfrac{7}{24}}$.

38. A man was born in a perfect cube year in the 18th century and died in a perfect square year in the same century. For how many years did the man live?

Solution: The 18th century is the 1700's. First, we find all the cubes between 1700 and 1799. $12^3 = 1728$ is the only valid cube. Next, we find all the squares in that interval. $42^2 = 1764$ is the only valid square. This gives the result $(1764 - 1728) = \boxed{36 \text{ years}}$ as the man's lifespan.

39. Kevin Koh is preparing a mixture of potassium hydroxide. If he has 1 liter of a solution of evenly mixed potassium hydroxide that has a total of 2 grams potassium hydroxide, find the amount of that solution he must add into another solution with 0.6 liters of water and no potassium hydroxide to get a final solution that has $\frac{4}{3}$ grams of potassium hydroxide?

Solution: The concentration by mass of the first solution is $\frac{2 \text{ grams}}{1 \text{ Liter}}$. We want to take x Liters of this solutions such that there are $\frac{4}{3}$ grams of potassium hydroxide in it; since the two concentrations must be the same, this means that $\frac{2 \text{ grams}}{1 \text{ Liter}} = \frac{4/3 \text{ grams}}{x \text{ Liter}}$. Cross-multiplying gives $2x = \frac{4}{3}$, and dividing both sides by two gives $x = \boxed{\dfrac{2}{3} \text{ Liters}}$.

40. How many integers from 1 to 100 are multiples of 2 and 3 but not 5?

Solution: Integers that are multiples of both 2 and 3 are multiples of $2 \cdot 3 = 6$. The smallest multiple of 6 between 1 and 100 is 6; the largest is 96. $6 = 1 \cdot 6$, and $96 = 16 \cdot 6$. Thus, there are 16 multiples of 6 between 1 and 100 $(1 \cdot 6, 2 \cdot 6, 3 \cdot 6, \ldots, 16 \cdot 6)$.

Integers that are multiples of 2, 3, and 5 are multiples of $2 \cdot 3 \cdot 5 = 30$. There are 3 multiples of 30 between 1 and 100 (30, 60, and 90). Since multiples of 30 are also multiples of 6, these 3 were included in our previous count of multiples of 6. Since we do not want multiples of 5, we must remove these 3. This leaves us with $\boxed{13}$ integers that are multiples of 2 and 3 but not 5.

41. Patricia is creating a daily schedule. She notices that she goes to school for one-third of the day, does homework for 12.5% of the day, plays tennis for three hours, and sleeps for the rest. Assuming she never does two things at once, what fraction of the day does Patricia spend sleeping?

Solution: Patricia does homework for $12.5\% = \frac{1}{8}$ of the day. She also plays tennis for $\frac{3 \text{ hours}}{24 \text{ hours}} = \frac{1}{8}$ of the day. In total, she spends $\frac{1}{3} + \frac{1}{8} + \frac{1}{8} = \frac{7}{12}$ of the day not sleeping. This means that she spends $1 - \frac{7}{12} = \frac{12-7}{12} = \boxed{\dfrac{5}{12}}$ of the day sleeping.

42. Alex is preparing for the BCA Math Competition. On the nth day, he does $2 \cdot n - 1$ problems. How many problems will Alex have done after the tenth day?

Solution: We can solve this problem by brute force. We simply have to compute $1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 = \boxed{100 \text{ problems}}$.

OR

Solution: We can also solve this problem through the use of an arithmetic series with first term 1 and common difference 2. In terms of d, the number of days, we will always get final answer d^2. It is left as an exercise to the reader to show this.

43. Sue has a set of ten numbers. How many ways can she pick eight of them?

Solution: Picking eight out of ten numbers means that she is not picking two numbers; thus, the problem is the same as saying that she picks two numbers. There are 10 choices for the first number, and (since one number has already been picked) 9 choices for the second number. So if order matters, there are $10 \cdot 9 = 90$ possible ways she can pick the two numbers. However, order does not matter. If she picks 1 and then 2, this is the same as picking 2 and then 1. Since this means that every pair will have appeared twice in our previous expression, we must divide by 2. Thus, there are $\boxed{45}$ possibilities.

OR

Solution:

$$\binom{10}{8} = \frac{10!}{8! \cdot (10-8)!} = \frac{10!}{8! \cdot 2!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdots 2 \cdot 1}{(8 \cdot 7 \cdot \cdots \cdot 2 \cdot 1) \cdot (2 \cdot 1)} = \frac{10 \cdot 9}{2} = 5 \cdot 9 = \boxed{45}$$

44. A circle with center O has points A and B on its circumference. The area of the circle is 36π and $\angle AOB = 90°$. Compute $\overline{AB}$.

Solution: Because AO and OB are radii, ΔAOB is a $45°, 45°, 90°$ triangle with hypotenuse AB. Furthermore, AO and OB have length 6 because the radius $= 6$ (Area $= \pi \cdot r^2$, so $36 = r^2$). This means, by the property of the special triangle, that $\overline{AB}$ equals $\boxed{6\sqrt{2}}$.

OR

Solution: Calculate the length of AO and BO as above. By the Pythagorean Theorem for right triangles, $(\overline{AB})^2 = (\overline{AO})^2 + (\overline{BO})^2 = 6^2 + 6^2 = 36 + 36 = 72 = 36 \cdot 2$, so $\overline{AB} = \sqrt{36 \cdot 2} = \sqrt{36} \cdot \sqrt{2} = \boxed{6\sqrt{2}}$

45. Yi is playing a game. He starts at the point $(1, 0)$. First he must run to the y-axis, then he must run to the point $(3, 4)$. What is the least distance that he has to run?

Solution: This is a seemingly easy problem but it is actually rather challenging. We can divide the problem into two parts: first getting to the y-axis, then getting to the point (3,4). Let us pretend that Yi has a "shadow" which runs with him on the first section but then, on the second section, does exactly what he does but on the opposite of the y-axis (the shadow becomes his reflection across the y-axis). It should be obvious that the shadow, in the course of its journey, will travel the same distance as Yi and will go from the point (1,0) to (-3,4), hitting the y-axis at the same time Yi does. Because the shortest distance between two points is a line, we must calculate the length of the line from (1,0) to (-3,4) to find the minimum distance Yi's shadow can run. The minimum distance is $\sqrt{(-3-1)^2 + 4^2} = \sqrt{(-4)^2 + 4^2} = \sqrt{16 + 16} = \sqrt{16 \cdot 2} = \sqrt{16} \cdot \sqrt{2} = \boxed{4\sqrt{2}}$, thus this is also the minimum distance that Yi must travel. Note that we could not simply let Yi himself travel on a straight line because that line would not pass through the y-axis.

46. Kevin drives from his home to his local library at thirty miles per hour. He makes the return trip at twenty miles per hour. In miles per hour, what is his average speed during the trip from his home to the library and back?

Solution: Let D be the distance from Kevin's home to the library in miles. Kevin spent $\frac{D}{30}$ hours on the first trip and $\frac{D}{20}$ hours on the return trip. The total length of the trip was $2D$ and the total time was $\frac{D}{30} + \frac{D}{20} = \frac{5D}{60} = \frac{D}{12}$. This makes the average speed equal $2D \div \left(\frac{D}{12}\right) = 2D \cdot \left(\frac{12}{D}\right) = 2 \cdot 12 = \boxed{24}$.

47. If three six-sided dice are rolled, what is the probability that their sum is 17?

Solution: The only way for the three dice to sum to 17 is if one of the rolls is 5 and the other two are 6, since $2 \cdot 6 + 5 = 17$. The ways to do this are (5,6,6), (6,5,6), and (6,6,5), so there are three ways to roll a sum of 17. The total possible rolls is $6 \cdot 6 \cdot 6 = 6^3 = 216$, so the probability of rolling a sum of 17 is $\frac{3}{216} = \boxed{\frac{1}{72}}$.

48. There are lights A,B,C,D,E,F,G in this order in a line. Every light has a switch. Right now lights A,C,E,G are on and the rest are off. Starting at A and walking to G, Alex turns the switch of each light once. He repeats this process again and again from A to G. After exactly 1999 switches have been turned, which lights are on?

    Solution: After 14 switches, each light will have been switched twice and so the lights will be in their original states. The remainder when 1999 is divided by 14 is 11, so the result of 1999 switches is indentical to the result of 11 switches. By brute force, one can see that switches A, B, C, and D will be switched twice, maintaining their state, and switches E, F, and G will be switched only once, altering their state. This means that $\boxed{\text{lights A, C, and F will be on}}$.

49. If $x^2 + 1 = 2 \cdot x$, then what is x?

    Solution: Subtracting $2x$ from both sides gives $x^2 - 2x + 1 = 0$; factoring the left-hand side gives $(x-1)^2 = 0$. Since $x - 1$ must be 0, this means that $\boxed{x = 1}$.

50. Jim sells 140 tickets for \$2001, some at full price and others half-price. Tickets sell for a whole number of dollars. How much money is raised by half-price tickets?

    Solution: Let F be the number of full price tickets and H the number of half price tickets. Also, let D be the cost of a full-price ticket. The following equations hold:

$$F + H = 140 \tag{1}$$

$$F \cdot D + \frac{H \cdot D}{2} = 2001 \tag{2}$$

    Multiply equation (1) by $\frac{D}{2}$ to get $\frac{F \cdot D}{2} + \frac{H \cdot D}{2} = 70 \cdot D$. Subtracting from equation (2), we find that $\frac{F \cdot D}{2} = 2001 - 70 \cdot D$, or $F \cdot D = 4002 - 140 \cdot D$, or $D \cdot (F + 140) = 4002$. Clearly D must be a divisor of 4002 between $\frac{4002}{140}$ and $\frac{4002}{280}$ (which means it is between 15 and 28). The prime factorization of 4002 is $2 \cdot 3 \cdot 23 \cdot 29$, so the only valid value of D is 23.

    Solving for H, we find that $H = 106$. $\frac{H \cdot D}{2}$ equals $\boxed{\$1,219}$, the total revenue generated by half price tickets.