7th Annual Bergen County Academies Math Competition

Eighth Grade

Sunday, 18 October 2009

1. In square meters, what is the area of a rhombus with both diagonals measuring $10\sqrt{2}$ cm?

   Solution: If the diagonals of a rhombus are equal in length, the rhombus must be a square. Let s be the side length of the square in centimeters. By the Pythagorean theorem, $s^2 + s^2 = (10\sqrt{2})^2 = 200W$, so $s^2 = 100$, so $s = 10$. Thus, each side length of the square is 10 cm long, or 0.1 m long. Thus, the area of the square in meters is $0.1^2 = \boxed{0.01 \text{ m}^2}$.

2. Sherry and Jenny are trying to find each other but they both have a terrible sense of direciton. Sherry starts out being exactly ninety meters west of Jenny. She walks thirty meters north, turns and walks eighty meters west, then turns again and walks ten meters south. Jenny walks fifty meters south, then turns and walks seventy meters east. How far apart in meters are they now?

   Solution: We may place Sherry at $(-90, 0)$ and Jenny at $(0, 0)$. Sherry then moves to $(-90, 30)$, then $(-170, 30)$ and finally $(-170, 20)$. Jenny moves to $(0, -50)$, then $(70, -50)$. The distance between the two is
   $$\sqrt{(-170-70)^2 + (20-(-50))^2} = \sqrt{240^2 + 70^2} = 10\sqrt{24^2 + 7^2} = \boxed{250}.$$

3. How many solutions does the equation $x = \sqrt{16}$ have?

   Solution: $\sqrt{16} = 4$, by definition. It is *not* -4; $\sqrt{x}$, where x is positive, is defined to be the unique positive real number y such that $y^2 = x$. Thus, our equation has exactly $\boxed{1}$ solution.

4. The average of Steve's five tests is 99. The average of Bill's three tests is 17. What average must Eric get on his two tests such thatthe average of the ten tests is at least a 73?

   Solution: The sum of Steve's five test scores is 495. The sum of Bill's three tests is 51. Collectively, the sum of their 8 tests is 546, but we want the collective sum to be 730. The difference of these two numbers is 184, which must be accomplished by Eric over two tests. Thus he must average 92 on his tests.

5. Point B lies on line $\overline{AC}$ such that B is nine units away from C and eleven units away from A. If the length of $\overline{AC}$ is an integer, how many possible values are there for the length of $\overline{AC}$?

   Solution: By the Triangle Inequality, we know that $AB + BC > AC$, so $AC < 20$. Again by the Triangle Inequality, we know that $AC + BC > AB$, so $AC + 9 > 11$ and thus $AC > 2$. Therefore, the number of possible values for AC is equal to the number of integers between 2 and 20 non-inclusive, and this number is simply $20 - 2 - 1 = \boxed{17}$.

6. How many integers from 1 to 100 inclusive are relatively prime with 140?

   Solution: $140 = 2^2 \cdot 5 \cdot 7$, so we merely seek the number of integers from 1 to 100 inclusive that do not have 2, 5, or 7 as prime factors. We will use complementary counting, and count the number of integers from 1 to 100 inclusive that do have 2, 5, or 7 as prime factors. There are 50 multiples of 2, 20 multiples of 5, and 14 multiples of 7 less than 100. There are 10 multiples of $5 \cdot 2 = 10$, 2 multiples of $5 \cdot 7 = 35$, and 7 multiples of $2 \cdot 7 = 14$ less than 100, and there is one multiple of $2 \cdot 5 \cdot 7 = 70$ less than 100. Thus, by the principle of inclusion-exclusion, our answer is $50 + 20 + 14 - 10 - 2 - 7 + 1 = \boxed{66}$ numbers between 1 and 100 inclusive that are relatively prime with 140.

7. Kelvin Wang sneaks into an intergalatic convention. There are two species of people in there: dinasauri (singular dinosaurus) and Yahaos. Given that each dinosaurus has seven heads, each Yahao has fifteen heads, and Kelvin correctly counted a total of eighty-two heads, find the total number of beings taking part in this convention, excluding Kelvin.

   Solution: Let there be d dinosauri and y Yahaos. We have that $7d + 15y = 82$. Since y and d are nonnegative integers, we need $y < 6$. Otherwise, $7d + 156 \geq 15y > 90 > 82$, which is impossible. Checking $y = 1, 2, 3, 4, 5$, it is easy to see that $y = 5$ and $d = 1$ is the only possible solution, so our answer is $y + d = 5 + 1 = \boxed{6}$.

8. Suppose a right triangle has a hypotenuse of length 3. Given that its area is 1, find its perimeter.

   Solution: Let a and b be the lengths of the legs of the triangle. Since the triangle's area is 1, we have $\frac{ab}{2} = 1$, i.e., $ab = 2$. Since its hypotenuse is 3, we have $a^2 + b^2 = 9$. We add $2ab$ to this, so $a^2 + 2ab + b^2 = 13$. Since $(a + b)^2 = a^2 + 2ab + b^2$, we have that $a + b = \sqrt{13}$. The perimeter of this triangle is $a + b + c = 3 + a + b = \boxed{3 + \sqrt{13}}$.

9. Evaluate $20_3 + 2_3 + 0.2_3 + 0.02_3$. Express your answer as a fraction in base 10.

   Solution: This is 22.22_3, which is $100_3 - 0.01_3$. $100_3 = 9$, and $0.01_3 = \frac{1}{9}$. Thus, $22.22_3 = 9 - \frac{1}{9} = \boxed{\frac{80}{9}}$.

10. Mike Sun has a floating chair that moves at 3 mph in water with no current. He gets onto his floating chair, and facing north, rides a current going south at 1 mph. After x minutes, he turns his chair around 180° and rides the same current going south. He notices that he arrives at his starting point exactly one hour after he first left. Find x.

    Solution: Observe that x is the amount of time Mike spent going north, and $60 - x$ is the amount of time he spent going south. When going north, Mike Sun moves at a speed of 2 mph. Thus, he covers a distance of $2x$ while facing north. When going south, Mike Sun moves at a speed of 4 mph, so he covers a distance of $4(60 - x)$ whe ngoing south. Since those distances must be the same, we have $2x = 240 - 4x$, so $6x = 240$, meaning that $\boxed{x = 40}$.

11. At what time between 4:00 and 5:00 does the minute hand and the hour hand overlap on an analog clock? Round your answer to the nearest minute.

    Solution: At 4:00, the hour hand makes a 120 degree angle with the minute hand. Each minute, the minute hand advances $\frac{360^\circ}{60} = 6^\circ$, while the hour hand advances $\frac{360^\circ}{12} \cdot \frac{1}{60} = 0.5^\circ$. Thus, each minute, the difference between the two angles decreases by 5.5°. Suppose m minutes have elasped when the minute hand and hour hand overlap. Since the difference between their two angles is 0, we have $120 - 5.5m = 0$, that is, $m = \frac{120}{5.5}$, so $m \approx 22$. Thus, 22 minutes have elapsed, so the time is $\boxed{4:22}$.

12. A triangle has three positive integral sides, $x - 10$, x, and $x + 10$. How many such triangles are obtuse?

    Solution: We first observe such a triangle exists if and only if the length of the longest side is shorter than the sum of the lengths of the other two sides (triangle inequality), that is, $x + 10 < x + x - 10$, that is, $x > 20$. We now observe that a triangle is obtuse if and only if the square of its longest side is shorter than the sum of the squares of the shorter two sides, i.e., $(x + 10)^2 > x^2 + (x - 10)^2$, that is, $x^2 + 20x + 100 > 2x^2 - 20x + 100$, that is, $x < 40$. In other words, a triangle with the three positive integral sides $x - 10$, x, and $x + 10$ is obtuse if and only if $20 < x < 40$. The number of integers between 20 and 40 exclusive is 40 - 20 - 1 = 19, so our answer is $\boxed{19}$.

13. Kevin solves ten math problems on the first day. Each day he increases the number of math problems solved per day by three problems (so on the second day he solves thirteen problems, sixteen problems on the third day, etc). Today is Monday. By the end of Sunday, how many problems would he have solved?

Solution: We seek $10 + 13 + 16 + 19 + 22 + 25 + 28 = \boxed{133}$.

14. Let x be the answer to this problem. What is $x^2 - 21x + 121$?

    Solution: The answer to this problem is x, so $x^2 - 21x + 121 = x$, i.e., $x^2 - 22x + 121 = (x - 11)^2 = 0$, so $\boxed{x = 11}$.

15. Let $f(x) = \frac{x^3-3x+2}{x^3-7x+6}$. Compute the sum of all distinct values of x for which $f(x) = 0$.

    Solution: Observe that $x^3 - 3x + 2 = (x - 1)^2(x + 2)$, and $x^3 - 7x + 6 = (x - 1)(x - 2)(x + 3)$. $f(x) = 0$ exactly when f is defined and its numerator is equal to 0, i.e., when $(x - 1)^2(x + 2) = 0$ and $x \neq 1, 2, -3$. If $(x - 1)^2(x + 2) = 0$, and $x \neq 1$, we get that $x = -2$ is our only solution, so the sum of all solutions is $\boxed{-2}$.

16. Alex Zhu's bus number is 534. He then noticed that it had three consecutive digits (3, 4, and 5, though not not necessarily in that order.) Bored, he correctly computes the total number of 3-digit numbers that contain three consecutive positive digits. Find that number.

    Solution: There are 7 possible sets of 3 consecutive positive digits, namely, $\{1, 2, 3\}$, $\{2, 3, 4\}$, $\ldots$, $\{7, 8, 9\}$. There are $3! = 6$ ways to arrange each of the elements of each set, so our answer is $6 \cdot 7 = \boxed{42}$.

17. How many integers from 1 to 1000 inclusive are multiples of 2 and 3 but not 5?

    Solution: The number of integers between 1 and 1000 inclusive that are multiples of 2 and 3 is the number of multiples of 6 between 1 and 1000 inclusive; there are $\lfloor \frac{1000}{6} \rfloor = 166$ such numbers. However, we have included multiples of 6 and 5, i.e., multiples of 30 in our count, when we should not have. Hence, we have overcouned $\lfloor \frac{1000}{30} \rfloor = 33$ numbers, giving a final answer of $166 - 33 = \boxed{133}$.

18. Evaluate $\left(-\frac{1}{2}\right)^{-1^{100}}$.

    Solution: $-1^{100} = -1$. Thus, $\left(-\frac{1}{2}\right)^{-1^{100}} = \left(-\frac{1}{2}\right)^{-1} = \frac{1}{-\frac{1}{2}} = \boxed{-2}$.

19. Let $f(x) = ax^2 + bx + c$, where a, b, and c are unknown real numbers. Given that $f(1) = 15$, $f(2) = 24$, and $f(3) = 35$, compute $f(6)$.

    Solution: We write the given information as $a + b + c = 15$, $4a + 2b + c = 24$, and $9a + 3b + c = 35$. Subtracting the first equation from the other two, we get $3a + b = 9$ and $8a + 2b = 20$, that is, $4a + b = 10$. Subtracting $3a + b = 9$ from $4a + b = 10$, we get $a = 1$. Thus, $b = 6$, and $c = 8$, so $f(x) = x^2 + 6x + 8$. Thus, $f(6) = \boxed{80}$.

20. Two sides of a right triangle are 3 and 4. Find all possible areas of the triangle.

    Solution: We have three scenarios: 3 and 4 are the lengths of the legs of the triangles, 4 is the length of the hypotenuse of the triangle and 3 is the length of one of the legs, and 3 is the length of the hypotenuse of the triangle and 4 is one of the legs. However, we can quickly eliminate the last possibility, as the hypotenuse is always the longest side in a right triangle. In the first case, the area is clearly $\frac{3 \cdot 4}{2} = 6$. In the second case, the Pythagorean theorem gives us that the length of the other leg is $\sqrt{7}$. The area of the triangle in the second case is $\frac{3\sqrt{7}}{2}$, so our two possible areas are $\boxed{6, \dfrac{3\sqrt{7}}{2}}$.

21. If x and y are real numbers satisfying $x^2 + y^2 = 1$, find the greatest possible value of $x + y$.

    Solution: Clearly, $x + y$ will be optimized when both x and y are positive. Otherwise, if one of x and y, say x, is negative, we may replace x with $-x$, since $(-x)^2 + y^2 = 1$ and $-x + y > x + y$. Thus, we may apply QM-AM. By QM-AM, we have $\sqrt{\frac{1}{2}} = \sqrt{\frac{x^2+y^2}{2}} \geq \frac{x+y}{2}$, i.e., $x + y \leq \frac{2}{\sqrt{2}} = \sqrt{2}$. Equality holds when $x = y = \frac{\sqrt{2}}{2}$, so our answer is $\boxed{\sqrt{2}}$.

OR

As before, we may assume that x and y are both positive. Graphically, we want to find the greatest possible k such that the graphs of $x^2+y^2=1$ and $x+y=k$ intersect. This is because we may simply let x and y be the given intersection point, for which we attain $x^2+y^2=1$ and $x+y=k$. (We may think of $x^2+y^2=1$ as the unit circle centered at the origin, and $x+y=k$ a family of lines with slope -1). In other words, we seek the line that is as far away as possible from the unit circle as possible while still intersecting it at least once, i.e., the one tangent to the unit circle. If it is indeed tangent, by symmetry, the coordinates of the tangency point will be (x,x) for some x; since $x^2+x^2=1$, we get $x=\frac{\sqrt{2}}{2}$ (since x lies in the first quadrant, as we assumed all varibles are positive.) Hence, our answer is $\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}=\boxed{\sqrt{2}}$.

OR

$(x+y)^2=x^2+y^2+2xy=1+2xy\leq 1+2xy+(x-y)^2=1+2xy+x^2-2xy+y^2=1+x^2+y^2=2$. Therefore, $x+y\leq\boxed{\sqrt{2}}$; this is attained when $x=y=\frac{\sqrt{2}}{2}$.

22. Let i be the square root of -1. Evaluate $i^{1^2}+i^{2^2}-i^{3^2}-i^{4^2}+i^{5^2}+i^{6^2}-i^{7^2}-i^{8^2}+...+i^{2009^2}+i^{2010^2}$.

    Solution: Note that for some integer n, i^n is 1 if $4|n$ and i if $4|n-1$. Observe now that when n is odd, $4|n^2-1$, and when n is even, $4|n^2$. Thus, $i^{(4k+1)^2}+i^{(4k+2)^2}-i^{(4k+3)^2}-i^{(4k+4)^2}=i+1-i-1=0$. We add this identity up for $k=0,1,\dots,1002$, and see that the sum of all but the last two terms in our expression is 0. Thus, we seek the sum of $i^{2009^2}+i^{2010^2}$, which is just $\boxed{1+i}$.

23. What is the minimum distance from the point $(0,0)$ to a point on the line $5x+12y=60$?

    Solution: Let $A=(0,5)$ $B=(12,0)$, and $O=(0,0)$. First observe that A and B lie on L. The minimum distance fom a point on L to O is clearly the foot of the altitude from O to L, i.e., the foot of the altitude from O to segment AB. Let h be this altitude. By the Pythagorean theorem, $AB=13$, so the area of ABO is $\frac{13h}{2}$. But the area of ABO is also $\frac{OA\cdot OB}{2}=\frac{5\cdot 12}{2}=30$. Thus, $\frac{13h}{2}=30$, so $h=\frac{60}{13}$, so our answer is $\boxed{\dfrac{60}{13}}$.

24. If a number has more than one digit, when we take the sum of its digits, it decreases. Therefore, if we constantly repeat the process of adding the digits of the number, we ultimately end up with a single digit. For any integer x, denote this single digit by $f(x)$. Find $f(6^{5^4})$.

    Solution: Let $s(n)$ denote the sum of the digits of n. It is well-known that $9|n$ if and only if $9|s(n)$, i.e., $9|s(s(n))$, etc. Thus, if $9|n$, $9|f(n)$. Since $9|6^{5^4}$, $9|f(6^{5^4})$. Since $f(6^{5^4})$ has only one digit, it must be that $\boxed{f(6^{5^4})=9}$.

25. If a circle centered at (1, 2) is tangent to the line $y=-x-3$, what is the radius of the circle?

    Solution: In order for the circle to be tangent to the line, the distance from the center of the circle to the line must be the circle's radius. To find this distance, we translate the circle so that it is centered at the origin. The line's new equation becomes $y+2=-(x+1)-3$, or $y=-x-6$. Its x-intercept and y-intercepts are $(-6,0)$ and $(0,-6)$, respectively; call them A and B, respectively. Let O be the origin, and let the distance from the O to the line $y=-x-6$ be h. By the Pythagorean theorem, $AB=6\sqrt{2}$. The area of $AOB=\frac{ABh}{2}$. But the area of AOB is also just $\frac{6\cdot 6}{2}=18$. Thus, $ABh=36$, so $h=\frac{36}{6\sqrt{2}}=3\sqrt{2}$. Thus, the radius of the circle must be $\boxed{3\sqrt{2}}$.

26. Peter runs two laps around a track. He runs the first lap in 30 yards per minute. He wants to set his goal for his average pace of the two laps to x yards per minute. What is the smallest x such that he can never achieve his goal?

Solution: Suppose that one lap around the track covers d yards and takes t minutes to cover. We have that $\frac{d}{t} = 30$, the rate at which he runs. We want to find the smallest x such that the equation $x = \frac{2d}{\frac{d}{30} + \frac{d}{r_2}}$ has no solution in r_2, where r_2 is positive. (The numerator represents the distance required to cover two laps; the demoniator represents the time required to cover two laps.) Rearranging, we get that $\frac{60-x}{30x} = \frac{1}{r_2}$. Clearly, when $x = 60$, there is no solution for r_2. Otherwise, we have $r_2 = \frac{30x}{60-x}$, so if Peter runs at that rate, he can attain his goal Therefore, $\boxed{60}$ is the smallest x such that he can never achieve his goal.

27. Two real numbers a and b are such that $a = 4b$ and $\log(a) + \log(b) = \log(a + b)$. Find $a \cdot b$.

    Solution: If $\log a + \log b = \log(a + b)$, since $\log(ab) = \log a + \log b$, we have $\log(ab) = \log(a + b)$, so $ab = a + b$. Since $a = 4b$, we have $4b^2 = 4b + b = 5b$. Since $b > 0$, $4b = 5$, so $b = \frac{5}{4}$, so $a = 4b = 5$. Thus, $\boxed{ab = \frac{25}{4}}$.

28. Suppose that $A \cdot B \cdot C + D \cdot E - F = 28$, where A, B, C, D, E, and F are the digits 1 through 6 with no digit repeated. If $A > B > C$ and $D > E$, what is the six-digit number $ABCDEF$?

    Solution: We first observe that it is necessary that F be even. Otherwise, since $A \cdot B \cdot C + D \cdot E = 30 + F$, it must then be that all the even digits are among A, B, and C; otherwise, $A \cdot B \cdot C$ and $D \cdot E$ would be even, making $30 + F$ even, a contradiction. But then, $A \cdot B \cdot C = 48$; $30 + F \leq 35$, since $F \leq 5$, so since $48 > 35$, this equation cannot hold, so F must be even.

    Suppose now that $F = 2$. Then $A \cdot B \cdot C + D \cdot E = 30$. 6 and 3 cannot both be among $A \cdot B \cdot C$ or $D \cdot E$, because then, the left-hand side would not be divisible by 3. Similarly, 6 and 4 cannot both be among the same products. Thus, 3 and 4 must be among the same products. If they are among D and E, then we have $A \cdot B \cdot C = 18$, but since 5 is among A, B, and C, this is impossible. If they are among $A \cdot B \cdot C$, we have that $C = 1$ or $C = 5$ (since $C \neq 6$, since 4, 3, and 6 are in different groups.) In the first case, we have $D \cdot E = 18$, a contradiction since 5 must divide $D \cdot E$, but 5 does not divide 18. In the second, we have that $A \cdot B \cdot C = 60 > 30$, again a contradiction, so $F \neq 2$.

    If $F = 4$, we again see that 2 and 6 cannot both be among A, B, C or D, E. In addition, 6 and 3 must both be among A, B, C or D, E; otherwise, $3|32$, a contradiction. If 6 and 3 are among D, E, then $A \cdot B \cdot C = 14$, but since $5|A \cdot B \cdot C = 14$, this is impossible. Thus, 6 and 3 are among A, B, C. It follows that $C = 1$. Otherwise, $A \cdot B \cdot C \geq 2 \cdot 18 = 36 > 32$, an impossibility. But then, $D, E = 5, 2$, so we have $18 + 10 = 32$, which is not true, so $F \neq 4$.

    Thus, it must be that $F = 6$. By arguments similar to those before, 2 and 4 cannot both be among A, B, C or D, E.If $A = 4$ and $D = 5$, then $E = 2$, so $B = 3$ and $C = 1$; it is easy to see this does not satisfy the equation. Thus, if $A = 4$, then $D \leq 3$. Thus, $A \cdot B \cdot C \leq 4 \cdot 3 \cdot 2 = 24$, and $D \cdot E \leq 3 \cdot 2 = 6$, so their sum is less than or equal to 30, an impossibility. If $A = 3$, then $B = 2$ and $C = 1$, so $D = 5$ and $E = 4$, which again does not satisfy the condition. Thus, $A = 5$. If $B = 4$, then $C = 1$; otherwise, $A \cdot B \cdot C \geq 2 > 34$, an impossibility. But then, $D = 3$ and $E = 2$; this does not satisfy the equation. If $B = 2$, again, $C = 1$, so $D = 4$ and $E = 3$. Once again, this does not satisfy the equation. Thus, $B = 3$. It follows that $C = 2$ (otherwise, $D = 4$ and $E = 2$, an impossibility), so $D = 4$ and $E = 1$. This satisfies our equation, so we have one solution, $\boxed{532416}$.

29. Chan has twenty-seven unit cubes, which he makes into a 3 by 3 by 3 cube. He paints the exterior red. He throws the cube into the air and it splits into twenty-seven cubes that are all resting on the ground. What is the expected number of red faces that are visible?

    Solution: There are 54 painted faces. After they are all on the ground we expect $\frac{5}{6}$ of the painted faces to be visible, because the ground covers $\frac{1}{6}$ of the faces. Hence, our answer is $54 \cdot \frac{5}{6} = \boxed{45}$.

30. What is the remainder when 2^{32} is divided by 25?

Solution: $2^{32} \equiv 4^{16} \equiv 16^8 \equiv (-9)^8 \equiv 81^4 \equiv 6^4 \equiv 36^2 \equiv 11^2 \equiv 21 \pmod{25}$, giving us an answer of $\boxed{21}$.

31. Pavel slices a perfectly spherical orange with a 2-inch radius. He notices that when the oranges are placed on the freshly-cut surfaces, one piece is one inch tall and the other is three inches tall. What is the radius (in inches) of the freshly-cut surface?

Solution: Let ω be the freshly-cut surface. Clearly, it is a circle. Let O be the center of the orange, let C be the center of ω, and let r be the radius of ω. Let A and B be the points where OC intersects the sphere, with A closer to C and B closer to O. (Loosely speaking, AB forms a "pole" through the sphere.) We are given that $AC = 1$ and $CB = 3$, so $AB = 4$. Thus, the diameter AB of the sphere has length 4, so the sphere's radius is 2. Since $AC = 1$ and $AO = 2$, we have $OC = 1$. Let P be any point on ω. The radus of the orange is 2, so $OP = 2$. Since $OC = 1$, by the Pythagorean theorem, $OC = \sqrt{3}$. But OC is the radius of ω, so our answer is $\boxed{\sqrt{3}}$.

32. Find the sum of all positive integers n for which $\frac{10n+77}{n+1}$ is also an integer.

Solution: $\frac{10n+77}{n+1} = 10 + \frac{67}{n+1}$. Thus, $\frac{10n+77}{n+1}$ is an integer if and only if $\frac{67}{n+1}$ is an integer, i.e., $n+1|67$. Since 67 is prime, it must be that $n + 1 = 67$ or $n + 1 = 1$; the latter case is impossible, since then $n = 0$. Thus, $n = 66$, so the sum of all possible values of n is $\boxed{66}$.

33. In hexagon $ABCDEF$, all angles between adjacent sides are equal. If $AB = 8$, $CD = 15$, and $DE = 17$, compute AF.

Solution: Extend $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, $\overline{EF}$, and $\overline{FA}$. Suppose $\overline{FA}$ and $\overline{CB}$ intersect at X, $\overline{BC}$ and $\overline{ED}$ intersect at Y, and $\overline{DE}$ and $\overline{AF}$ intersect at Z. Since hexagon $ABCDEF$ is equilangular, all angles between adjacent sides are $120°$. It follows that $m\angle BAX = m\angle ABX = m\angle YCD = m\angle YDC = m\angle ZFB = m\angle ZEF = 60°$, so $\triangle BAX$, $\triangle DCY$, and $\triangle EFZ$ are equilateral. In addition, $m\angle AXB = m\angle CYD = m\angle EZF = 60°$, so $\triangle XYZ$ is also equilateral. Since $AB = 8$, $AX = BX = 8$. Since $DC = 15$, $CY = DY = 15$. Since $\triangle XYZ$ is equilateral, we have $XZ = YZ$, that is, $XF + FZ = YE + EZ$. Since $\triangle FEZ$ is equilateral, $FZ = EZ$, so $XF = YE$, i.e., $AF + AX = ED + DY$, that is, $AF + 8 = 17 + 15$. Thus, $\boxed{AF = 24}$.

34. What is the largest n such that the difference $100! - 99!$ is divisible by 10^n? ($n!$ is $n \cdot (n-1) \cdot \ldots \cdot 1$)

Solution: $100! - 99! = 99!(100 - 1) = 99 \cdot 99!$. Since 10 and 99 are relatively prime, we merely seek the largest n such that $10^n|99!$. Note that this is the same thing as asking for the largest power of 5 dividng 99!. This number is just the sum of the number of multiples of 5 less than 99 and the number of multiples of 25 less than 99, which is $\lfloor \frac{99}{5} \rfloor + \lfloor \frac{99}{25} \rfloor = \boxed{22}$.

35. Mike was looking at the numbers one day and he stumbled upon the number 236. He decided to call this number a "growing" number, because the integers increased in value strictly from left to right. How may four-digit growing numbers are there?

Solution: For any set of four distinct digits $1--9$, there is exactly one way to rearrange them so that they are in increasing order, i.e., they can be arranged to form a growing number. Likewise, we can correspond to any growing number a set of four distinct digits 1-9, so the number of growing numbers is exactly the number of sets of four distinct digits from 1-9, which is just $\binom{9}{4} = \boxed{126}$.

36. Find all values of x for which $3x^2 - 39x + 126 < 0$.

Solution: The inequality is equivalent to the one we get when we divide through by 3, i.e., $x^2 - 13x + 42 < 0$, that is, $(x - 6)(x - 7) < 0$

Let $f(x) = (x - 6)(x - 7)$. If $x = 6$ or $x = 7$, $f(x) = 0$. If $x > 7$, we have that $x - 7, x - 6 > 0$, and therefore $f(x) = (x - 7)(x - 6) > 0$. If $x < 6$, then $x - 7, x - 6 < 0$, so $-(x - 7), -(x - 6) > 0$, so

$f(x) = (x-7)(x-6) > 0$. If $6 < x < 7$, then $x - 6 > 0$ and $x - 7 < 0$, so $f(x) = (x-7)(x-6) < 0$. Thus, our solution set is exactly the set of all real numbers x such that $\boxed{6 < x < 7}$.

37. Victoria and Weili are competing in a hot dog eating contest. Because he fasted for a week, Weili has a $\frac{4}{5}$ chance of winning a match and Victoria has a $\frac{1}{5}$ chance of winning a match (there are no ties). What is the probability that Victoria will win in at least two matches and at most four matches if they compete for five matches?

    Solution: We wish to find the probability that Victoria wins n out of the 5 matches. Victoria has probability $\frac{1}{5}$ of winning a given match and probability $\frac{4}{5}$ of losing, so the probability that she wins n matches is $\left(\frac{1}{5}\right)^n \left(\frac{4}{5}\right)^{5-n}$. However, we can order these matches in multiple ways, so we must also choose the n matches which Victoria wins. This can be done in $\binom{5}{n}$ ways, so our final probability is $\binom{5}{n}\left(\frac{1}{5}\right)^n \left(\frac{4}{5}\right)^{5-n} = \binom{5}{n}\left(\frac{4^{5-n}}{5^5}\right)$. We must add these quantities for $n = 2, 3, 4$; our final answer is

$$\frac{(10)(4^3) + (10)(4^2) + (5)(4^1)}{5^5} = \frac{820}{5^5} = \boxed{\frac{164}{625}}$$

    .

38. Find the smallest positive integer n such that 7 divides $\underbrace{11\ldots1}_{\text{n}}$

    Solution: Observe that $7|\frac{10^n-1}{9}$ if and only if $7|10^n - 1$; thus, we seek the smallest n such that $7|10^n - 1$. Now, observe that $7|10^n - 1 \iff 7|(7+3)^n - 1 \iff 7|3^n - 1$ (this is true because of the binomial theorem.) Thus, we seek the smallest n such that $7|3^n - 1$. We can easily guess and check to see that $\boxed{n = 6}$.

    OR

    As before, $7|\frac{10^n-1}{9}$ iff $7|10^n - 1$. We seek the smallest n such that $10^n \equiv 1 \bmod 7$, i.e., $3^n \equiv 1 \bmod 7$. By Fermat's little theorem, $3^6 \equiv 1 \bmod 7$. Thus, if $3^n \equiv 1 \bmod 7$, where n is minimal, $n|6$ (a property of orders of elements.) Since $3^1, 3^2, 3^3 \not\equiv 1 \bmod 7$, it follows that $\boxed{n = 6}$.

39. Jordan and Ian arrange a meeting to talk about Star Wars. Jordan plans to arrive at a random time between 1 o'clock PM and 4 o'clock PM, and will wait for an hour for Ian. Ian, on the other hand, plans to arrive at a random time between 12 o'clock PM and 5 o'clock PM, but will wait onlny 30 minutes for Jordan. What are the chances that the two meet to talk about Star Wars?

    Solution: Ian will be present for 30 minutes at some time from 12 to 5:30. Jordan will be present some time between 1 and 5 and will be there for an hour. Because 1 to 5 is contained entirely inside 12 and 5:30, it suffices to focus on the time that Jordan spends on the court. The earliest Ian can arrive and still meet Jordan is 30 minutes before Ian arrives. The latest Ian can arrive is the moment in which Jordan is just leaving. Thus, there is a 90 minute interval in which Ian can arrive, and it must be entirely in the 300 minute interval in which she can possibly arrive. Therefore, our final answer is $\frac{90}{300} = \boxed{\frac{3}{10}}$.

40. Which of the following can be factored into two quadratic polynomials with integer coefficients? $A: x^4 + 1, B: x^4 + 4, C: x^4 + 9, D: x^4 + 16, E: x^4 + 25$

    Solution: Let a be any integer. We have $x^4 + a^2 = (x^2 + a)^2 - 2ax^2 = (x^2 - \sqrt{2a}x + a)(x^2 + \sqrt{2a}x + a)$, by difference of squares. Since polynomials have unique factorization, it is therefore necessary that both $x^2 - \sqrt{2a}x + a$ and $x^2 + \sqrt{2a}x + a$ both have integer coefficients, i.e., $\sqrt{2a}$ is an integer. Among $a = 1, 2, 3, 4, 5$, only $a = 2$ makes $\sqrt{2a}$ an integer, i.e., only $x^4 + 4$ can be factored into two quadratic polynomials with integer coefficients, so our answer is $\boxed{B}$.

41. Triangle $\triangle ABC$ has sides $AB = 6$, $BC = 8$, and $CA = 10$. There is a point P such that it is distance R from A, B, and C. What is πR^2?

    Solution: Line PB splits right angle $\angle ABC$ into two angles, $\angle PBA$ and $\angle PBC$. $\angle PBA$ has measure $90 - a$ and $\angle PBC$ has measure a. Notice that the triangles $\triangle ABP$, $\triangle BPC$, and $\triangle CPA$ are isosceles. Calculating the measures of the other angles in terms of a gives $\angle PAB = \angle PBA = 90 - a$, $\angle BPA = 180 - \angle PBA - \angle PAB = 2a$, $\angle PCB = \angle PBC = a$, $\angle BPC = 180 - \angle PBC - \angle PCB = 180 - 2a$, and $\angle CPA = 360 - \angle CPB - \angle BPA = 180$. Therefore, CPA is a line and is equivalent to CA. Also, because $R = CP = PA = \frac{CA}{2} = 5$, the value of πR^2 is 25π.

42. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. If x is a randomly chosen number between 0 and 42, compute the probability that $\lfloor 17x \rfloor = 17\lfloor x \rfloor$.

    Solution: Let $x = m + k$, where m is an integer and $0 \leq k < 1$. Then $m = \lfloor x \rfloor$. We wish to find the probability that $\lfloor 17m + 17k \rfloor = 17\lfloor m + k \rfloor$. Clearly, for any integer n, $\lfloor n + x \rfloor = n + \lfloor x \rfloor$. Since m and $17m$ are integers, our equation translates to $17m + \lfloor 17k \rfloor = 17m + 17\lfloor 17k \rfloor$, that is, $\lfloor 17k \rfloor = 17\lfloor k \rfloor$. But since $0 \leq k < 1$, $\lfloor k \rfloor = 0$, so the right-hand side of our equation is 0. Thus, $\lfloor 17k \rfloor = 0$, so $0 \leq 17k < 1$, so $0 \leq k < \frac{1}{17}$. Since x was arbitrarily chosen from the interval $(0, 17)$, the probability that k is any particular real number between 0 and 1 is the same. Thus, our answer is $\boxed{\dfrac{1}{17}}$.

43. If the equation $x^2 + 4x + y^2 + 6y = m$ has exactly one pair of real solutions (x, y), find m.

    Solution: We have $x^2 + 4x + 4 + y^2 + 6y + 9 = m + 4 + 9$, i.e., $(x + 2)^2 + (y + 3)^2 = m + 13$. Since $(x + 2)^2 + (y + 3)^2 \geq 0$, $m + 13 \geq 0$, i.e., $m \geq -13$. If $m > 13$, the equation is the graph of a circle with radius $\sqrt{m + 13}$; thus, there are infinitely many solutions. When $m = -13$, we get that $(x + 2)^2 = (y + 3)^2 = 0$, i.e., $x = -2$ and $y = -3$. Thus, when $m = -13$, there is indeed exactly one solution. Thus, $\boxed{m = -13}$.

44. Yumi is playing a computer game when she notices her score is a three-digit palindrome. She notices she is 382 points away from another palindrome score. What is the highest score she could have at that moment?

    Solution: To get our maximum as high as possible, we first see if it is possible that the first digit of the palindrome is 9. Then our palindrome is $\overline{9A9}$ for some digit A. We have that $\overline{9A9} + 382$ is again a palindrome; it must be in the form $\overline{1BB1}$, where B is also a digit. When we add these up, we note that $B = 2$ or $B = 3$, since $9 + 3$ ends in a 2, and there is a possibility that there was a carry from $A + 8$. But if $B = 2$, then we have that our original palindrome has to be $1221 - 382 = 839$, which is not possible. Thus, $B = 3$, so our original palindrome is $1331 - 382 = 949$; this satisfies our conditions, and is the only 3-digit palindrome that starts with 9, so our answer must be $\boxed{949}$.

45. Suppose we have a dartboard with radius 1. Suppose an equilateral triangle is inscribed in the dartboard. Find the probability that when a dart is thrown, it lands inside the triangle.

    Solution: The area of the dartboard is clearly π. We now seek the area of the triangle; the probability will then be the ratio of the area of the triangle to the area of the circle. Let O be the center of the dartboard, and let A, B, and C be the vertices of the triangle. Clearly, $[OAB] = [OBC] = [OCA]$, where $[XYZ]$ denotes the area of $\triangle XYZ$. We now seek $[OAC]$. Let H be the foot of the altitude from O to AC. Since $\angle AOC = 120°$ and $\angle CAB = 60°$, and HO and AO bisect $\angle COA$ and $\angle CAB$, respectively, we see that HAO is a 30-60-90 triangle. Thus, $OH = \frac{\sqrt{3}}{2}$ and $AH = \frac{1}{2}$, so $[AHO] =$

$\frac{OH \cdot AH}{2} = \frac{\sqrt{3}}{8}$. Thus, $[AOC] = 2[AHO] = \frac{\sqrt{3}}{4}$, so $[ABC] = 3[AHO] = \frac{3\sqrt{3}}{4}$. Thus, our desired probability is $\boxed{\frac{3\sqrt{3}}{4\pi}}$.

46. The polynomial $x^5 + x + 1$ can be expressed in the form $(x^2 + ax + b)(x^3 + cx^2 + dx + e)$, for some integers a, b, c, d, and e. Compute $a + b + c + d + e$.

Solution: $x^5+x+1 = x^5-x^2+x^2+x+1 = x^2(x^3-1)+x^2+x+1 = x^2(x-1)(x^2+x+1)+x^2+x+1 = (x^2+x+1)(x^3-x^2+1)$. Since polynomials factor uniquely, we have $a = 1, b = 1, c = -1, d = 0, e = 1$. Thus, $a + b + c + d + e = 1 + 1 - 1 + 1 = \boxed{2}$.

OR

Expanding, we get $(x^2 + ax + b)(x^3 + cx^2 + dx + e) = x^5 + x^4(a + c) + x^3(d + ac + b) + x^2(e + ad + bc) + x(ae + bd) + be$. Equating coefficients, we get that $a + c = 0$, $d + ac + b = 0$, $e + ad + bc = 0$, $ae + bd = 1$, and $be = 1$.

From $be = 1$, we get that either $b = e = 1$ or $b = e = -1$. If $b = e = -1$, then $ad = bc$ from our fourth equation. Since $a = -c$ from the first equation, we have $b = -d$ or $a = c = 0$. If $a = c = 0$, then $e = 0$ from the third equation, an impossibility, so $b = -d$. From the second equation, we have $ac = 0$, so $a = 0$ or $c = 0$, but from the first equation, we have $a + c = 0$, so $a = c = 0$, a contradiction.

Thus, $b = e = 1$. From the fourth equation, we have $a + d = 1$; from the third, we have $1 + ad + c = 0$; from the second, we have $1 + ac + d = 0$. Since $d = 1 - a$, we have $2 - a + ac = 0$, so $a(1 - c) = 2$. Thus, $a = 1$ and $d = 0$, and $c = -1$, or $a = -1, c = 3$, and $d = 2$. But the second solution set does not satisfy $1 + ad + c = 0$, while the first does. Thus, $a = 1$, $b = 1$, $c = -1$, $d = 0$, and $e = 1$. Adding these up, we have $a + b + c + d + e = 1 + 1 - 1 + 0 + 1 = \boxed{2}$.

OR

Let $\omega^3 = 1$, $\omega \neq 1$. Observe that $\omega^5 + \omega + 1 = \omega^2 + \omega + 1 = 0$. Since ω is a root of $x^2 + x + 1$, all roots of $x^2 + x + 1$ are also roots of $x^5 + x + 1$. Thus, $x^2 + x + 1 | x^5 + x + 1$. Let $Q(x) = \frac{x^5+x+1}{x^2+x+1}$, so $x^5 + x + 1 = (x^2 + x + 1)Q(x)$. We therefore have $a = 1, b = 1$. We now seek $c + d + e$. Note that this is $1+c+d+e-1 = Q(1)-1$, since the sum of the coefficients of Q is simply $Q(1)$.$Q(1)-1 = \frac{3}{3} = 1-1 = 0$, so our sum $a + b + c + d + e = a + b = \boxed{2}$.

47. Suppose a triangle $\triangle ABC$ has sides $\overline{AB} = \sqrt{73}$, $\overline{BC} = 10$, and $\overline{CA} = 9$ such that $\overline{AD} = \overline{DE} = \overline{EC}$. Point F is the midpoint of BC. Let P be the intersection of lines $\overline{BE}$ and $\overline{DF}$. If $\overline{BD}$ is perpendicular to $\overline{AC}$, what is the sum of the areas of $\triangle BPF$ and $\triangle DPE$?

Solution: Note that E is the midpoint of DC and that F is the midpoint of BC, so P is the centroid of $\triangle BDC$. Thus, $[PDE] = [PBF] = \frac{[BDC]}{6} = \frac{\frac{6 \cdot 8}{2}}{6} = 4$. Thus, $[PDE] + [PBF] = \boxed{8}$.

48. Suppose $f(x) = x^2 - 10x + 28$. Find the two integer solutions to the equation $f(f(x)) = x$.

Solution: Note that $f(x) = x$ implies that $f(f(x)) = f(x) = x$. $f(x) = x$ means that $x^2-10x+28 = x$, that is, $(x - 4)(x - 7) = 0$, giving $\boxed{x = 4, x = 7}$ as our integer solutions. (It can easily be verified with polynomial division that the other two roots of the polynomial are not integers; nonetheless, this should be of no concern to the person doing this problem. When he/she finds two integer solutions, it is necessary that those two are the answers to this question, so they don't need to worry about the other two roots.)

49. A scientist develops a procedure to test for a terrible disease, which occurs in 1% of the population. If the subject is indeed diseased, the test will be report this 95% of the time. However, even if a subject is healthy, the test will give a false positive x% of the time. If a subject tests positive on the test, there

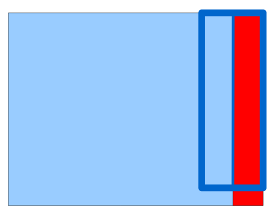
is a 50% chance that this subject has the disease. What is x, rounded to two decimal places? If x were 100%, write in 100.

Solution: For any event A, denote $\neg A$ to be the event not A, denote $P(A)$ to be the probability of event A occurring, denote $P(A|B)$ to be the probability of A occuring given that B has already happened, and denote $P(A \cap B)$ to be the probability that both A and B have occurred. It is fairly intuitive to see that $P(A|B)P(B) = P(A \cap B)$: the probability that both A and B occur is the probability that B occurs multiplied by the probability that A occurs, given that B has already occurred. From this, we see that $P(A|B)P(B) = P(A \cap B) = P(B|A)P(A)$ for any events A and B.

Let D denote the event that someone actually has the disease, and let $+$ denote the event that someone has tested positive. The problem statement tells us that $P(D) = 0.01$, $P(+|D) = 0.95$, $P(+|\neg D) = x\%$, and $P(D|+) = 0.5$. We have that $P(D)P(+|D) = P(+)P(D|+)$, that is, $0.01 \cdot 0.95 = 0.5 \cdot P(+)$, so $P(+) = 0.019$. Note that $1 - P(D|+) = P(\neg D|+) = 0.5$, and $P(\neg D) = 1 - 0.01 = 0.99$. Therefore, $P(+|\neg D)P(\neg D) = P(\neg D|+)P(+)$, so $x\% \cdot 0.99 = 0.5 \cdot 0.019$. Therefore, $x\% = \frac{0.5 \cdot 0.019}{0.99} = 0.0096$ to two decimal places, so $\boxed{x = 0.96}$.

OR

Solution:



Consider the above diagram of the problem (not drawn to scale). The red shaded region represents the 1% of the population that has the disease; the blue shaded region is the 99% that doesn't. The blue-outlined sub-rectangle represents the people who tested positive; 95% of the red region is in this box, as is x% of the blue region. Finally, since testing positive means that there is a 50% of actually having the disease, half of the blue-outlined rectangle is in the red region, and half of it is in the blue region. There are two ways of finding the areas of these halves. Since the red region is $1\% = 0.01$ of the entire population, and the right half is $95\% = 0.95$ of the red region, the right half is $0.01 \cdot 0.95$ of the entire population. Similarly, the blue region is $99\% = 0.99$ of the entire population, and the left half is $x\% = \frac{x}{100}$ of the blue region, and so the left half is $\dfrac{0.99x}{100}$ of the entire population. Since the two halves are equal, we have $0.01 \cdot 0.95 = \dfrac{0.99x}{100}$, or

$$x = \frac{100 \cdot 0.01 \cdot 0.95}{0.99} = \frac{95}{99} = 0.\overline{95} \approx \boxed{0.96}.$$

50. Suppose the sequence of numbers $a_0, a_1, \ldots, a_n$ are defined so that $2^{a_k} = 2^{2^k} + 1$. What is the integer nearest to $a_0 + a_1 + \ldots + a_6$?

Solution: Note that $a_k = \log_2(2^{2^k}+1)$. Hence,

$$\begin{aligned}
a_0 + a_1 + \ldots + a_6 &= 0 + \log_2(2^{2^0}+1) + \log_2(2^{2^1}+1) + \ldots + \log_2(2^{2^6}+1) \\
&= \log_2(2^{2^0}-1) + \log_2(2^{2^0}+1) + \log_2(2^{2^1}+1) + \ldots + \log_2(2^{2^6}+1) \\
&= \log_2((2^{2^0}-1)(2^{2^1}-1)) + \log_2(2^{2^1}+1) + \ldots + \log_2(2^{2^6}+1) \\
&= \log_2(2^{2^1}-1) + \log_2(2^{2^1}+1) + \log_2(2^{2^2}+1) + \ldots + \log_2(2^{2^6}+1) \\
&= \log_2(2^{2^2}-1) + \log_2(2^{2^2}+1) + \ldots + \log_2(2^{2^6}+1) \\
&= \\
&\vdots \\
&= \log_2(2^{2^7}-1) \\
&= \log_2(2^{128}-1) \\
&\approx \boxed{128}
\end{aligned}$$