2015 Joe Holbrook Memorial Math Competition
6th Grade Exam Solutions

The Bergen County Academies Math Team

October 11th, 2015

1. $7 + 9 + 10 + 10 + 7 + 2 + 2 + 1 + 1 + 1 = \boxed{50}$.

2. By the order of operations, we multiply before adding so $3 \times 5 + 4 \times 2 = 15 + 8 = \boxed{23}$.

3. $5 \times 6 = \boxed{30}$.

4. The Sun sets every evening, which comes after noon. Thus between noon Saturday and noon Thursday, Saturday, Sunday, Monday, Tuesday, and Wednesday have evenings; this makes $\boxed{5}$ days.

5. There are 11 fish, each of which eats 19 flakes, so the total amount that Nemo needs can be computed by $11 \times 19 = \boxed{209}$.

6. All multiples of 5 end in a 0 or a 5. This means that all numbers ending in a 1 or a 6 have a remainder of $\boxed{1}$.

7. The first term of the sequence is 1, and each subsequent term is 3 more than the previous one, so the second term is $1+3 = 4$, the third term is $1+3+3 = 7$, and so on. Extending this, the nth term is $1+3(n-1)$, so the 100th term is $1+3(100-1) = 1+3(99) = 1+297 = \boxed{298}$

8. $3(\mathbf{3(3+2)+2(3+2)}) + 2(\mathbf{3(3+2)+2(3+2)})$
   $= (3+2)(3\mathbf{(3+2)} + 2\mathbf{(3+2)})$
   $= (3+2)((3+2)(3+2)) = 5^3 = \boxed{125}$.

9. The least positive multiple of 7 is $7 \times 1 = 7$, while the greatest less than 100 is $7 \times 14 = 98$. Thus there are $\boxed{14}$ multiples.

10. If there are 2015 consecutive integers, then they must be $n-1007, n-1006, \cdots n-1, n, n+1, \cdots n+1006, n+1007$. When computing the sum, each $n-k$ is canceled out by $n+k$, so the total sum is $2015n$ and the mean is n, which is also the median of the list. The mean is 2, so $\boxed{2}$ is the median as well.

11. $f = 4b \rightarrow \frac{f}{4} = b$. $8b = 3p \rightarrow 48p = 16 \cdot 3p = 16 \cdot 8b = 16 \cdot \frac{8b}{4} = \boxed{32}b$

12. We keep track of the candy after each exchange:

Time	Andrew	Jackson
1	17	14
2	17+3	14
3	20	14-2
4	20-5	12+5
5	15	17

The positive difference between the two is $17 - 15 = \boxed{2}$.

13. There are three ways to choose the cheese. Once Thomas has chosen the cheese, he now has four options for his cracker. Thus for each of the three cheeses, there are four cracker options, so the number of combinations is $3 \times 4 = \boxed{12}$.

14. Notice that this is asking for the LCM of 4, 5, and 6. First take 4 and 6. It is easy to see that the LCM is 12. Now, consider 5 and 12. Since they are relatively prime, their LCM is simply their product, which is $\boxed{60}$

15. Recognizing that each number is one more than the one before it, we can group the terms into pairs: $0 + (-1 + 2) + (-3 + 4) + \cdots + (-99 + 100) = 0 + 1 + 1 + \cdots + 1$. Each positive even number ends a pair, and there are 50 positive even integers less than 100, so the final sum is $\boxed{50}$

16. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} = \frac{6}{12} + \frac{8}{12} + \frac{9}{12} = \boxed{\frac{23}{12}}$.

17. First, we compute what Matt pays. His discount is $10\% \times \$20 = 0.1 \times \$20 = \$2$, so he pays \$18 total. Tanny pays $\$25 = \$20 + \$5$, so his 15% discount applies only to \$20: $15\% \times \$20 = 0.15 \times \$20 = \$3$ is his discount, so he pays in total \$22. Thus, the answer is $\boxed{\text{Tanny, \$22}}$.

18. Alex catches up by $\frac{1}{2}$ of a foot every second. Since he needs to catch up on what is initially a 20 foot lead, he will take $\boxed{40}$ seconds to do this.

19. $2^1 = \mathbf{2}, 2^2 = \mathbf{4}, 2^3 = \mathbf{8}, 2^4 = 1\mathbf{6}$, and $2^5 = 3\mathbf{2}$. Because there is no carrying over to the units digit, this pattern repeats with a length of four. 245 is 1 more than a multiple of four, so the last digit of 2^{245} is the same as that of 2^1, which is $\boxed{2}$.

20. $111111^2 = 12345654321$. $s(12345654321) = \boxed{36}$.

21. If the product of two integers is odd, then both of them must have been odd. This means that the probability of the product being odd is the same as the probability of having rolled two odd numbers. The probability of rolling an odd number for a single roll is $\frac{1}{2}$, and dice rolls are independent events, so we can compute the probability by multiplying: $\frac{1}{2} \times \frac{1}{2} = \boxed{1/4}$.

22. Simplifying the fraction and grouping some terms, we see that $(23 + 46 \times 23 + \frac{46}{2}) - 48 \times 23 = (48 \times 23) - 48 \times 23 = \boxed{0}$.

23. Three-digit numbers are greater or equal to 100 and less than 1000. The least multiple of 17 greater than 100 is $17 \cdot 6 = 102$, and the greatest less than 1000 is $7 \cdot 58 = 986$. This means there are $58 - 6 + 1 = \boxed{53}$ multiples.

24. There is one way to select no toppings, and one way to select all of them. There are three ways to choose one topping, and the number of ways to choose two toppings is equivalent to choosing one topping to *not* put onto the ice cream, of which there are three ways. Thus the number of ways is $1 + 1 + 3 + 3 = \boxed{8}$.

25. A square has four equal sides, so its perimeter is four times a side length, i.e. $4 \times 18 = 72$. A hexagon has six sides, so each side is $\frac{1}{6}$ of the perimeter, which is same as the square's, so the side length is $\frac{72}{6} = \boxed{12}$.

26. Note that 10 Dragons = 25 YoungGuys, and 10 Dragons = 75 Puzzle, so 25 YoungGuys = 75 Puzzle, or 1 YoungGuy = $\boxed{3}$ Puzzle.

27. Let's relabel the letters slightly: JHM_1M_2C. Since they are all different, there are $5! = 120$ arrangements. But in our original word, the M's were the same. Since in a given arrangement, there are $2! = 2$ ways to arrange them, we divide out by this factor of overcounting, to get $\boxed{60}$ arrangements.

28. Suppose Kelvin's roll is written as an ordered pair, with the first number coming from the first die. Then, the rolls that obtain a 5 are $(1,4)$, $(2,3)$, $(2,3)$ (duplicated because 2 appears twice on the first one), $(4,1)$. There are 4 desired outcomes out of $6 \times 6 = 36$ total possible outcomes, for a final answer of $\frac{4}{36}$, or $\boxed{\frac{1}{9}}$.

29. There are 10 questions in Part A and 15 questions in Part B, so there must be 25 questions total. In order to receive a score of at least 80%, Minsung must have gotten at least $\frac{80}{100} \cdot 25 = 20$ of the questions correct. He already has 7 questions right from Part A, which means that he needs to get at least $20 - 7 = \boxed{13}$ more to receive a score of at least 80%.

30. The sum of the first n positive integers is $\frac{n(n+1)}{2}$. $10 + 11 + 12 + \cdots + 99 + 100 = 1 + 2 + 3 + \cdots + 99 + 100 - (1 + 2 + 3 + \cdots + 9 + 10) = \frac{100(101)}{2} - \frac{10(11)}{2} = 5050 - 45 = \boxed{5005}$.

31. Switching digits will not reduce an integer if its first digit is greater than or equal to its second. Because of this the only two digit numbers that satisfy the condition with the first digit n are $nn, nn-1, \cdots, n0, n+1$ in total. Since a two digit number can only start with a number from 1 to 9, this means that there are $(1 + 1) + (2 + 1) + \ldots + (9 + 1) = 54$ valid two digit integers. And since there are 90 two digit integers, $\frac{3}{5}$ of them work.

32. $6^{12} + 12^6 = 6^6(6^6 + 2^6) = 12^6(3^6 + 1) = 12^6(730) = 10 \times 12^6(73) \implies \boxed{73}$.

33. Note that if $a_n = a_{n-1} + a_{n-2} + \cdots + a_0$ and $a_{n-1} = a_{n-2} + a_{n-3} + \cdots + a_0$, then $a_n = a_{n-1} + (a_{n-2} + a_{n-3} + \cdots + a_0) = a_{n-1} + a_{n-1} = 2a_{n-1}$. Repeating this argument, we can find that $a_n = 2^k n_{n-k}$ as long as $n - k \geq 3$. $a_3 = a_2 + a_1 + a_0 = 2 + 1 + 0 = 3$, so $a_8 = 2^5 a_{8-5} = 32a_3 = \boxed{96}$.

34. By observing the first few n^n, we see that the sequence grows quite quickly; $1^1 = 1, 2^2 = 4, 3^3 = 27, 4^4 = 256, 5^5 = 3125$, and so on. In fact, the subsequent one is near our upper limit of 50000: $6^6 = 46656$. $50000 - 46656 = 3344$, so our answer would be 6 if the sum of the previous n^n was less than 3344. However, $4^4 + 5^5 = 256 + 3125 = 3381$, which is too big, so our maximum n must be $\boxed{5}$.

35. Let the side of the cube be s. The volume is s^3, and the surface area is the sum of the areas of the six equal square faces, each of which has area s^2. Thus we have the equation $s^3 = 2(6s^2)$. Dividing both sides by s^2, we find that $s = \boxed{12}$.

36. Suppose that $A = 0$. If $X = \times$, then $AXB = 0$ regardless of the value of B, and the value of the expression $AXBYC$ is C, as Y is forced to be +. Thus when $A = 0$, $AXBYC = 1$ if $C = 1$ and 2 if $C = 2$. Analogously, when $C = 0$ the value is determined by A.
Now suppose that $B = 0$. If $X = \times$, then the value of A doesn't matter, and $AXBYC = C$. If $Y = \times$, then C doesn't matter, so $AXBYC = A$. These can only be 1 or 2, so there are $\boxed{2}$ possible values.

37. The probability that Boris will pitch well given he is pitching with his right hand is $\frac{4}{5}$. However, the probability that he is using his right hand at all is $\frac{5}{8}$, so we can multiply the two to find that the probability that a throw will be a good right-handed one will be $\frac{4}{5} \times \frac{5}{8} = \frac{1}{2}$. Similarly, the probability that a pitch will be good and left-handed is $\frac{4}{9} \times \frac{3}{8} = \frac{1}{6}$. We add the two for a total of $\boxed{\frac{7}{12}}$.

38. The external angle measure of a regular n-gon is $\frac{360}{n}$, so the external angle measure of a regular decagon is $\frac{360}{10} = 36°$. Meanwhile, the sum of the internal angles of an n-gon is $180(n-2)$; in an octagon, this is $180 \times 6 = 1080$. A regular polygon is equiangular, so the angle measure of a single angle in an octagon is $\frac{1080}{8} = 135°$. The difference between the two is $135° - 99° = \boxed{36°}$.

39. The sum of the digits is a multiple of 15, which means that it is also a multiple of 3, so the number itself is divisible by 3. It is also given that this number is a multiple of 35; since 35 is coprime to 3, the number must be a multiple of $3 \times 35 = 105$. Checking the three-digit multiples of 105, we see that $105 \times 7 = \boxed{735}$ satisfies the conditions.

40. If n has d digits, then on the list, n's digits will occupy nd spots.
Let's approach this using casework, starting with $d = 1$, i.e. $n = 1, \dots, 9$. These numbers occupy $1 + 2 + \cdots + 9 = 45$ spots.
When $d = 2$, the numbers $10, 11, \dots, n$ occupy

$$20 + 22 + \cdots + 2n = 2((1 + 2 + \cdots + n) - (1 + 2 + \cdots + 9)) = n(n+1) - 90$$

digits, so there have been a total of $n(n+1) - 45$ digits written down so far. We want to find the maximum n such that this no more than 2015: $n(n+1) - 45 \leq 2015 \implies n^2 + n - 2060 \leq 0 \implies n \leq 44$. This means that after we have finished with $n = 44$, we will have written $44 \times 45 - 45 = 1935$ digits. We begin to write $4545\cdots$, with 4's on even entries and 5's on odd ones. Thus, the 2015th digit will be a $\boxed{5}$.

41. Here we will utilize subtractive counting; instead of the probability that the coin will land heads at least twice, we will compute the probability that it will land heads less than two times, and subtract the result from 1. The probability that all flips are tails is $(\frac{1}{3})^4 = \frac{1}{81}$, while the probability that there is exactly one heads is $(\frac{1}{3})^3 \times \frac{2}{3} \times 4 = \frac{8}{81}$. Thus the probability that there will be less than two heads is $\frac{8}{81} + \frac{1}{81} = \frac{1}{9}$, so our answer is $1 - \frac{1}{9} = \boxed{\frac{8}{9}}$.

42. What is a_7? $a_7 = a_6 \times a_5 \times \cdots \times a_1 = 41$. $a_8 = a_7 \times a_6 \times a_5 \times \cdots \times a_1 = (a_7)^2 = \boxed{1681}$.

43. Given two similar figures, the ratio of the areas is the square of the ratio of the corresponding side lengths, which is equal to the ratio of the perimeter. Thus we have $\frac{18}{8} = R^2 \implies R = \frac{3}{2}$, where R is the ratio of the perimeters. The perimeter of the pentagon with area 8 is 6, which means that the perimeter of the larger one is $6R = 6 \times \frac{3}{2} = \boxed{9}$.

44. Suppose the prism's lengths are r, s, t. We are given $rs = 16$, $st = 20$ and $tr = 45$. We want rst. If we multiply the 3 equations given, we get $(rs)(st)(tr) = r^2s^2t^2 = (rst)^2 = (16)(20)(45) \implies rst = \boxed{120}$.

45. We begin our search by looking for numbers with exactly 20 factors, and throughout this solution we will use the method demonstrated here: $360 = 2^3 3^2 5^1$ has $(3+1)(2+1)(1+2) = 24$ positive factors.
$20 = 10 \times 2 = 5 \times 4 = 5 \times 2 \times 2$. Label these cases a, b, c and d, in that order.

 (a) The minimal case is 2^{19}. This is probably wrong.
 (b) The minimal case is $2^9 3 = 1536 << 2^{19}$. We can rule out case a.
 (c) The minimal case is $2^4 3^3 = 432$. This is better than b, but we will keep searching.
 (d) The minimal case is $2^4 3 \times 5 = 240$. This is the best so far.

Note that $240 < 256 = 2^8$. It is also possible that our number will have 21, 22 or even 23 factors. 24 and beyond are clearly not feasible, and in fact since 23 is prime, it is also less than ideal ($2^{22} >> 2^8 > 240$). Similarly, for 22, the only options are 2^{21} or $2^{10}3$, and for 21 are 2^{20} or $2^6 3^2$, all of which fail. Thus, the answer is $\boxed{240}$.

46. Substituting $a+b$ for c in the third equation, we have $a+b+d = a \implies b = -d$. Since b is positive, d must be negative. Substituting $-d$ for b in the second equation, $c = 2d$. We want to maximize $a+b+c+d = a+c$, so we want to minimize the magnitude of any negative addends. d is a negative integer, so the greatest possible value of c is $2(-1) = -2$. Returning to the first equation, we have $a = -2-b$. Since b is positive, the greatest value of a is when b is the least, i.e. $b = -1$. Thus the least possible value of $a = -2-1 = -3$, so the maximum possible value of $a+b+c+d$ is $-2-3 = \boxed{-5}$.

47. We know that his rolls are some permutation of $a, b, 2a$, with $a \leq b \leq 2a$. We quickly see that $a \leq 3$, so our set of tuples $(a, b, 2a)$ (possibly reordered) is reasonably sized. The possibilities are:

- $(1,1,2)$: 3 arrangements
- $(1,2,2)$: 3 arrangements
- $(2,2,4)$: 3 arrangements
- $(2,3,4)$: 6 arrangements
- $(2,4,4)$: 3 arrangements
- $(3,3,6)$. 3 arrangements
- $(3,4,6)$: 6 arrangements
- $(3,5,6)$: 6 arrangements
- $(3,6,6)$. 3 arrangements

In total, there are 36 rolls that satisfy the condition, out of 216 total possible rolls of the 3 die. Thus, the answer is $\boxed{\frac{1}{6}}$.

48. $2015 = 5 \times 13 \times 31$, $2015^{2015} = 5^{2015}13^{2015}31^{2015}$. Any factor of 2015^{2015} is of the form $5^a 13^b 31^c$, and for that factor to be a square, a, b, c, all have to be even. There are 1008 choices for each of a, b, c, so $N = 1008^3 = (2^4 3^2 7)^3 = 2^{12}3^6 7^3$. Perfect square factors of N are of the form $2^a 3^b 7^c$, where a, b, c are all even. There are 7 choices for a, 4 choices for b and 2 choices for c. Thus the desired value is $7 \times 4 \times 2 = \boxed{56}$.

49. It takes Rebecca $R(x,y) = \frac{|x|}{20} + \frac{|y|}{10}$ hours to reach point (x,y), and it takes The Great Bustard $GB(x,y) = \frac{\sqrt{x^2+y^2}}{10}$ hours to reach that same point. We wish to consider the set of points (x,y) with $GB(x,y) < R(x,y)$. We will only consider the first quadrant for the time being (as for $f = R$ or $f = GB$, $f(x,y) = f(-x,y) = f(x,-y) = f(-x,-y)$, so we can eliminate the $|\cdot|$ signs).
We want

$$\frac{x}{20} + \frac{y}{10} > \frac{\sqrt{x^2+y^2}}{10} \implies \frac{x}{2} + y > \sqrt{x^2+y^2} \implies \frac{x^2}{4} + xy + y^2 < x^2 + y^2 \implies xy > \frac{3}{4}x^2 \implies y > \frac{3}{4}x$$

We see that this region, in quadrant 1 and restricted to $x, y < 1$, has area $\frac{5}{8}$. A similar figure will occur in the other 4 quadrants, so the desired area is $\boxed{\frac{5}{2}}$.

50. Because M is the midpoint of the tangents to the zero circle at B and circle B, AM is the radical axis of the zero circle at B and circle ω_1. Hence, $AN \times AC = AB^2 = 16$. Then $AN = \frac{16}{9}$, so $NC = AC - AN = 9 - \frac{16}{9} = \frac{65}{9}$, so $|AN - NC| = \frac{65-16}{9} = \frac{49}{9}$.