2015 Joe Holbrook Memorial Math Competition
7th Grade Exam Solutions

The Bergen County Academies Math Team

October 11th, 2015

1. We have two 5's, two 7's, two 9's, two 2's, one 3, and one 1. This simplifies to $2 \cdot (5 + 7 + 9 + 2) + 4 = 2 \cdot (23) + 4 = 46 + 4 = \boxed{50}$. .

2. The number of water bottles he drinks in a day is the sum of the numbers of water bottles he drinks at each meal. $2 + 3 + 5 = \boxed{10}$.

3. All multiples of 5 end in a 0 or a 5. This means that all numbers ending in a 1 or a 6 have a remainder of $\boxed{1}$.

4. The Sun sets every evening, which comes after noon. Thus between noon Saturday and noon Thursday, Saturday, Sunday, Monday, Tuesday, and Wednesday have evenings; this makes $\boxed{5}$ days. .

5. The first term of the sequence is 1, and each subsequent term is 3 more than the previous one, so the second term is $1+3 = 4$, the third term is $1+3+3 = 7$, and so on. Extending this, the nth term is $1 + 3(n-1)$, so the 100th term is $1 + 3(100-1) = 1 + 3(99) = 1 + 297 = \boxed{298}$

6. If Sung Hyup was 15 years old 4 years ago, he is now 19 years old. Sunny is 3 years younger than him, so he is currently 16. In two years he will be $16 + 2 = \boxed{18}$

7. $0 + (-1 + 2) + (-3 + 4) + \cdots + (-99 + 100) = 0 + 1 + 1 + \cdots + 1$. Each positive even number ends a pair, and there are 50 positive even integers less than 100, so the final sum is $\boxed{50}$

8. Three-digit numbers are greater or equal to 100 and less than 1000. The least multiple of 17 greater than 100 is $17 \cdot 6 = 102$, and the greatest less than 1000 is $7 \cdot 58 = 986$. This means there are $58 - 6 + 1 = \boxed{53}$ multiples.

9. There are 12 integers from 3 to 14 inclusive, and 19 from 17 through 35. This makes for $\boxed{31}$ questions.

10. We can find the number of stickers he receives per week, then multiply that by three. On each of Monday and Tuesday, he receives 1 sticker; on Wednesday, 2 stickers; none on Thursday and Friday. This amounts to $1 + 1 + 2 + 0 + 0 = 4$ stickers per week, resulting in $\boxed{12}$ for the entire time period.

11. Let's track their progress using coordinates, setting x+ to be forward movement and y+ to be upward movement. Their path is as follows, in relative coordinates: (0,0), (0,10), (0.5,10), (0.5,2), (25.5,2), (25.5,11), (27,11), (27,1), (37,1). Their x-path has length 37. Their y-path is the sum of the absolute values of the individual y-differences: 10+8+9+10=37. Their total path has length 37+37=74.

12. A pyramid is the shape formed by a polygonal base and the line segments from each of the vertices of the base to another point in a different plane, called the *vertex* of the pyramid.

If the base is octagonal, then there are 8 edges on the octagon and 8 edges from each of the vertices, for a total of 16; there is the octoganal face, with 8 triangular faces formed by the vertex-edges and the edges of the octagon, for a total of 9. Thus the answer is $16 + 9 = \boxed{25}$.

13. The cafeteria and the gym would be the furthest apart if they were on opposite sides of the dorm; the distance would then be $20 + 10 = 30 hops$. Contrarily, the shortest distance would be if they were along a straight line on the same side of the dorm, and would equal $20 - 10 = 10$. Thus the final answer is $30 - 10 = \boxed{20}$.

14. Notice that this is asking for the LCM of 4, 5, and 6. First take 4 and 6. It is easy to see that the LCM is 12. Now, consider 5 and 12. Since they are relatively prime, their LCM is simply their product, which is $\boxed{60}$.

15. $a + b = 7 \implies a^2 + 2ab + b^2 = (a + b)^2 = 7^2 = 49$. We are given $a^2 + b^2 = 29$, so subtracting gives us $2ab = 20 \implies ab = \boxed{10}$.

16. The *triangle inequality* states that the sum of the lengths any two sides of a triangle are greater than the third. Think about it: if the sum was less than the longest side, then you couldn't close the triangle! Knowing this, we know that the longest length of the third side has to be less than $12 + 13 = 25$; the greatest such integer is $\boxed{24}$.

17. $2^1 = \mathbf{2}, 2^2 = \mathbf{4}, 2^3 = \mathbf{8}, 2^4 = 1\mathbf{6}$, and $2^5 = 3\mathbf{2}$. Because there is no carrying over to the units digit, this pattern repeats with a length of four. 245 is 1 more than a multiple of four, so the last digit of 2^245 is the same as that of 2^1, which is $\boxed{2}$.

18. $(7 * 13) * 23 = ((8 \cdot 7) - (3 \cdot 13) - 7) * 23 = 10 * 23 = (8 \cdot 10) - (3 \cdot 23) - 7 = 80 - 69 - 7 = \boxed{4}$.

19. Notice that AC is the diagonal of rectangle ABCD, so triangle ABC is a right triangle with side lengths 3 and 6 and hypotenuse AC. By the Pythagorean Theorem, $(AC)^2 = 3^2 + 6^2 = 9 + 36 = 45$; $AC = 3\sqrt{(}5)$.

20. The external angle measure of a regular n-gon is $\frac{360}{n}$, so the external angle measure of a regular decagon is $\frac{360}{10} = 36°$. Meanwhile, the sum of the internal angles of an n-gon is $180(n - 2)$; in an octagon, this is $180 \cdot 6 = 1080$. A regular polygon is equiangular, so the angle measure of a single angle in an octagon is $\frac{1080}{8} = 135°$. The difference between the two is $135° - 99° = \boxed{36°}$.

21. Let's relabel the letters slightly: $\text{JHM}_1\text{M}_2\text{C}$. Since they are all different, there are $5! = 120$ arrangements. But in our original word, the M's were the same. Since in a given arrangement, there are $2! = 2$ ways to arrange them, we divide out by this factor of overcounting, to get $\boxed{60}$ arrangements.

22. The smaller circle has diameter 2, so the smaller square has side length 2. Thus, the smaller square has diagonal length $2\sqrt{2}$. That diagonal forms the diameter of the larger circle, so its radius is $\sqrt{2}$, and the larger square has side length $2\sqrt{2}$. The smaller circle has area π. The larger circle has area $\pi \cdot \sqrt{2}^2 = 2\pi$, and the smaller square has area $2^2 = 4$, so the total desired area is $(2\pi - 4) + \pi = \boxed{3\pi - 4}$.

23. $5^{2015} = 25^{1007} * 5$. The remainder when 25 is divided by 24 is 1, so the remainder of any power of 25 divided by 24 must be 1. This can be seen if you expand the expression $(24 + 1)^k$; each addend except the 1 has a factor of 24, so the whole thing save the 1 is divisible by 24, so the remainder is 1. Multiplying this by the leftover factor of 5 gives an answer of $\boxed{5}$.

24. Note that if $a_n = a_{n-1} + a_{n-2} + \cdots + a_0$ and $a_{n-1} = a_{n-2} + a_{n-3} + \cdots + a_0$, then $a_n = a_{n-1} + (a_{n-2} + a_{n-3} + \cdots + a_0) = a_{n-1} + a_{n-1} = 2a_{n-1}$. Repeating this argument, we can find that $a_n = 2^k n_{n-k}$ as long as $n-k \geq 3$. $a_3 = a_2 + a_1 + a_0 = 2+1+0 = 3$, so $a_8 = 2^5 a_{8-5} = 32a_3 = \boxed{96}$.

25. Let the side of the cube be s. The volume is s^3, and the surface area is the sum of the areas of the six equal square faces, each of which has area s^2. Thus we have the equation $s^3 = 2(6s^2)$. Dividing both sides by s^2, we find that $s = \boxed{12}$.

26. By observing the first few n^n, we see that the sequence grows quite quickly; $1^1 = 1, 2^2 = 4, 3^3 = 27, 4^4 = 256, 5^5 = 3125$, and so on. In fact, the subsequent one is near our upper limit of 50000: $6^6 = 46656$. $50000 - 46656 = 3344$, so our answer would be 6 if the sum of the previous n^n was less than 3344. However, $4^4 + 5^5 = 256 + 3125 = 3381$, which is too big, so our maximum n must be $\boxed{5}$.

27. The digit in the seventh position is always removed, and there will always be a digit in the seventh position until there are less than seven digits total, i.e. when there are six; the first six digits are never touched, so the final number will be $\boxed{123456}$.

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29. $EFCD$ is a trapezoid. One base, CD, has length 6, and we are given that the other, EF, has length 4. The height is 6, because $E, F \in AB$, so the area is $\frac{b_1+b_2}{2}h = \frac{4+6}{2}6 = \boxed{30}$.

30. $6^{12} + 12^6 = 6^6(6^6 + 2^6) = 12^6(3^6 + 1) = 12^6(730) = 10 \times 12^6(73) \implies \boxed{73}$.

31. 30 is four less than a multiple of 17, which means that the number of 6's we add must sum to four more than a multiple of 17. The first time this occurs is $6 \cdot 12 = 72 = 17 \cdot 4 + 4$, so Kang Myung will write for 12 more days. There are three days left until the deadline, which means he will be $12 - 3 = \boxed{9}$ days late.

32. Here we will utilize subtractive counting; instead of the probability that the coin will land heads at least twice, we will compute the probability that it will land heads less than two times, and subtract the result from 1. The probability that all flips are tails is $\frac{1}{3}^4 = \frac{1}{81}$, while the probability that there is exactly one heads is $\frac{1}{3}^3 \cdot \frac{2}{3} \cdot 4 = \frac{8}{81}$. Thus the probability that there will be less than two heads is $\frac{8}{81} + \frac{1}{81} = \frac{1}{9}$, so our answer is $1 - \frac{1}{9} = \boxed{\frac{8}{9}}$.

33. We are given that $\frac{a+b+c}{3} = 46$. We want to find $\frac{(a-7)+(b+4)+(c+6)}{3} = \frac{a+b+c+3}{3} = \frac{a+b+c}{3} + 1 = \boxed{47}$. (It turns out that the sequence information was superfluous.)

34. The sum of the digits is a multiple of 15, which means that it is also a multiple of 3, so the number itself is divisible by 3. It is also given that this number is a multiple of 35; since 35 is coprime to 3, the number must be a multiple of $3 \cdot 35 = 105$. Checking the three-digit multiples of 105, we see that $105 \cdot 7 = \boxed{735}$ satisfies the conditions.

35. We will approximate these relative to each other. We note that

$$3^7 = 2187 > 2015. \text{ So } a = 3^{2015} > 3^{21} = (3^7)^3 > 2015^3 = b$$

We also note that $3^6 = 729 < 1009$.

$$c = 1009^{1009} > 1009^{336} > 729^{336} = (3^6)^{336} = 3^{2016} > 3^{2015} = a.$$

Thus, $\boxed{c > a > b}$.

36. Suppose the prism's lengths are r, s, t. We are given $rs = 16$, $st = 20$ and $tr = 45$. We want rst. If we multiply the 3 equations given, we get $(rs)(st)(tr) = r^2s^2t^2 = (rst)^2 = (16)(20)(45) \implies rst = \boxed{120}$.

37. We see that $AB = \sqrt{(-1-2)^2 + (3-7)^2} = \sqrt{25} = 5$. Call $a = BC, b = AC$, and suppose $b > a$. By the triangle inequality, $a + 5 > b \implies b - a < \boxed{5}$.

38. If n has d digits, then on the list, n's digits will occupy nd spots.
Let's approach this using casework, starting with $d = 1$, i.e. $n = 1, \ldots, 9$. These numbers occupy $1 + 2 + \cdots + 9 = 45$ spots.
When $d = 2$, the numbers $10, 11, \ldots, n$ occupy $20 + 22 + \cdots + 2n = 2((1 + 2 + \cdots + n) - (1 + 2 + \cdots + 9)) = n(n+1) - 90$ digits, so there have been a total of $n(n+1) - 45$ digits written down so far. We want to find the maximum n such that this no more than 2015: $n(n+1) - 45 \le 2015 \implies n^2 + n - 2060 \le 0 \implies n \le 44$. This means that after we have finished with $n = 44$, we will have written $44 \times 45 - 45 = 1935$ digits. We begin to write $4545\cdots$, with 4's on even entries and 5's on odd ones. Thus, the 2015th digit will be a $\boxed{5}$.

39. The first step is to recognize the impossible side lengths: 2-2-4, 2-2-6, and 2-4-6. There are 2 ways to achieve the first, 2 to achieve the second, and 8 for the third. Thus there are 12 ways to choose side lengths that would not create a triangle. There are $\binom{6}{3} = 20$ total ways to choose 3 cards, so the probability that we get a triangle is $1 - \frac{12}{20} = 1 - \frac{3}{5} = \boxed{\frac{2}{5}}$.

40. We begin our search by looking for numbers with exactly 20 factors, and throughout this solution we will use the method demonstrated here: $360 = 2^3 3^2 5^1$ has $(3+1)(2+1)(1+2) = 24$ positive factors.
$20 = 10 \times 2 = 5 \times 4 = 5 \times 2 \times 2$. Label these cases a, b, c and d, in that order.

   (a) The minimal case is 2^{19}. This is probably wrong.
   (b) The minimal case is $2^9 3 = 1536 << 2^1 9$. We can rule out case a.
   (c) The minimal case is $2^4 3^3 = 432$. This is better than b, but we will keep searching.
   (d) The minimal case is $2^4 3 \times 5 = 240$. This is the best so far.

   Note that $240 < 256 = 2^8$. It is also possible that our number will have 21, 22 or even 23 factors. 24 and beyond are clearly not feasible, and in fact since 23 is prime, it is also less than ideal ($2^{22} >> 2^8 > 240$). Similarly, for 22, the only options are 2^{21} or $2^{10}3$, and for 21 are 2^{20} or $2^6 3^2$, all of which fail. Thus, the answer is $\boxed{240}$.

41. We use Vieta's formulas to see that $a + b = 52$ and $ab = 365$. Thus, $Q(x) = x^2 - 365x + 52$, so again by Vieta's $c + d = ab = 365$ and $cd = a + b = 52$. So, $a + b + c + d = 52 + 365 = \boxed{417}$

42. Let F on BC be the point such that EF is perpendicular to BC. Notice that since $AD = DE$ and ASA congruency, triangles ADC and DEF are congruent. Also notice, then, that CD is congruent to FE, and the length of BE is twice the length of FE, since triangle BEF is right and angle B is 30 degrees. So, all we have to do is find the length of CD and multiply by 2. Let this length be X. $CD = X$. $DF = 1$, since DF is congruent to $AC = 1$. $FB = X \cdot \sqrt{3}$ because $FE = X$, and BEF is a $30-60-90$ triangle. Also, since BAC is also a $30-60-90$ triangle, BC has length $\sqrt{3}$. Let's add up the sides: $CD + DF + FB = CB$, so $X + 1 + X \cdot \sqrt{3} = \sqrt{3}$, so $X = \frac{(4-2\sqrt{3})}{2}$, so Our answer is $\boxed{4 - 2\sqrt{3}}$.

43. We see that $x^2 + 1 = 13x$ so $x + \frac{1}{x} = 13$. Let $f(n) = x^n - frac1n$ for n an integer.We see that $f(2n) = f(n)^2 - 2$. So $f(2) = 13^2 - 2 = 167$, and f(4)=$166^2 - 2$, which ends in $\boxed{7}$.

44. Note that the desired sum is equal to $S = (1 + \frac{1}{3} + \frac{1}{9} + \cdots) + (\frac{1}{3} + \frac{4}{9} + \frac{7}{27} + \frac{10}{81} + \cdots) = \frac{1}{1-\frac{1}{3}} + \frac{1}{3} + \frac{1}{3}S \implies \frac{2}{3}S = \frac{3}{2} + \frac{1}{3} = \frac{11}{6} \implies S = \boxed{\frac{11}{4}}$.

45. Suppose A_1A_2 lies horizontally. on the xyplane. Then each subsequent segment is either perpendicular to at a 45° angle with the axes. By the Pythagorean theorem, the x and y-components of each segment at 45° are $\frac{1}{\sqrt{2}}$ of the segment. Placing the nonagon on the coordinate plane with A_1 as the origin, we can find the coordinates of A_9 by adding all of the horizontal and vertical components, accounting for positive and negative directions. The x coordinate would be $1+\sqrt{2}+0+(-2\sqrt{2})+(-5)+(-3\sqrt{2})+0+4\sqrt{2} = -4$. Similarly, the y coordinate is $0 + \sqrt{2} + 3 + 2\sqrt{2} + 0 + (-3\sqrt{2}) + (-7) + (-4\sqrt{2}) = -4 - 4\sqrt{2}$. By the Pythagorean Theorem, the distance from $(-4, -4-4\sqrt{2})$ to $A_1 = (0,0)$ is $\sqrt{64 - 32\sqrt{2}}$, so the answer is $64 - 32 = \boxed{32}$.

46. Using the Law of Cosines, we can find the distance, d between the two hands, given the angle θ between them: $d = \sqrt{2^2 + 3^2 - 2(2)(3)cos\theta}$. Plugging in $d \leq \sqrt{7}$, we have $7 \geq 13 - 12cos\theta \implies -60° < \theta < 60°$, i.e. $\theta \in (0°, 60°) \cup (300°, 360°)$ gives a desired value of d. At 1 o'clock, the hands form a 330° angle; the angle cannot be any larger at any previous point. We know that the minute hand moves continuously away from the hour hand, so every real degree measure from 0° to 330° is met. This means 90° out of 330° possible angles works, which means our final answer is $\frac{90}{330} = \boxed{\frac{3}{11}}$.

47. Let $Z(x) = d(x-1)(x-2)(x-3) + ax^2 + bx + c$. Now remainder when $Z(x)$ is divided by $(x-1)(x-2)(x-3)$ is $ax^2 + bx + c$. $Z(1) = b + c + a = 2, Z(2) = 4a + 2b + c = 12, Z(3) = 9a + 3b + c = 28$. Giving us solution of, $a = 3, b = 1, c = -2$, therefore answer is $\boxed{1}9$

48. Define $A = (2,3)$, $B = (26,3)$, and $C = (14,19)$. Our third point is somewhere on triangle ABC. First, notice that triangle ABC is isosceles with $AB = 24$ and $AC = BC = 20$. The expected area of our desired triangle is $\frac{bh}{2}$,or the product of 5 and the y-coordinate of our third point. So, let's set up a weighted system for each y-coordinate. The probability that our third point is on AB (and thereby have y-coordinate of 3) is 24/64, or 3/8, since $AB = 24$ and the perimeter is 64. So, we now have $3 \cdot (\frac{3,8}{)}$. If our third point is NOT on AB (with probability $\frac{5}{8}$), each y-coordinate between 3 and 19 has an equal chance of being chosen, so the expected y-coordinate value is at the midpoint; i.e. 11. So now we have $11 * (\frac{5}{8})$. Adding these up gives us $(3 \cdot (\frac{3}{8})) + (11 \cdot (\frac{5}{8})) = 8$, so our expected height is 8, so our expected area is $5 * 8 = \boxed{40}$.

49. Note that $-a + b + c = 1 - 2a$, and analogously for b and c. Now, we multiply through by $(1-2a)(1-2b)(1-2c)$ to yield $(1-2a)(1-2b) + (1-2b)(1-2c) + (1-2c)(1-2a) = (1-2a)(1-2b)(1-2c) \implies 1-2a-2b+4ab+1-2b-2c+4bc+1-2c-2a+4ca = 1-2a-2b-2c+4ab+4bc+4ca-8abc \implies -2a-2b-2c+2 = -8abc \implies a+b+c-1 = abc \implies abc = \boxed{0}$.

50. Consider $f(x) = k$ for some real k. then $\frac{(x^2+4x+13}{(3x^2+2x+3)} = k$ must have solution. If this expression is re-written $(3k-13) \cdot x^2 + (2k-4)x + (3k-1) = 0$, since this has a solution, discriminant must be positive. $(k-2)^2 - (3k-1)(3k-13) \geq 0$ which simplifies to $8k^2 - 2k + 9 \leq 0$ therefore, the two solution extremes of the solution multiply to $\boxed{\frac{9}{8}}$ by using vieta's formula