2015 Joe Holbrook Memorial Math Competition
8th Grade Exam

The Bergen County Academies Math Team

October 11th, 2015

Instructions

DO NOT TURN OVER THIS PAGE UNTIL YOU ARE TOLD TO DO SO.

The Joe Holbrook Memorial Math Competition is a 90-minute, 50-question exam. Each question has exactly one correct answer; writing two different answers, EVEN IF ONE IS CORRECT, is worth no credit. In addition, **only answers written on the answer sheet provided will be graded.**

Be advised that proctors cannot answer any questions about terminology, notation or the questions.

On the JHMMC, you may only use writing utensils, erasers, and scrap paper provided by the proctors. You MAY NOT use calculators, compasses, protractors, straightedges, or your own scrap paper.

All answers must be fully simplified and exact. For example:

- Write $2\sqrt{2}$, rather than $2.83\ldots$
- Write $\frac{4}{3}$, rather than $1\frac{1}{3}$ or $1.33\ldots$

In addition, know that $[A_1A_2\cdots A_n]$ denotes the area of n-gon $A_1A_2\cdots A_n$.

1. What is $1+3+6+8+10+8+6+4+2+2$?

2. Moonbeam is sixteen years old, while Sunshine is twenty. Of the two, who was born second?

3. Thomas the Mouse wants to eat cheese and crackers. He has 3 kinds of cheese and 4 kinds of crackers. How many ways are there to choose 1 cheese and 1 cracker?

4. After returning home from school, Zack wants to watch his favorite TV show, *The Adventures of Super Matthew*. If each episode is twenty minutes long, and Zack gets home at 5:00, how many complete episodes can he watch before he must begin his homework at 7:29?

5. What is $0-1+2-3+4-5+\cdots-99+100$?

6. Two sides of a triangle have lengths 12 and 13. What is the largest possible integer side length of the third side?

7. Jungle Jim, the proprietor of the jungle gym, is charging admission. If the admission fee for Matt is \$20, but he has a coupon for a 10% discount, and the admission fee for Tanny is \$25, but he has a coupon for a 15% discount on every dollar after \$5 that he pays, who pays more, and how much does he pay?

8. What is the least product one could obtain by multiplying two numbers in the set $\{-7,-5,-1,1,3\}$?

9. When I add 9 to my favorite number and triple the result, I get the fourth power of the smallest odd prime number. What is half of my favorite number?

10. Esther rolls two fair dice each with faces numbered 1 through 6. What is the probability that the product of the two rolled numbers is odd?

11. What is the sum of the possible perimeters of an isosceles triangle whose side lengths include 7 and 15?

12. The 400-digit number $1234567812345678\ldots12345678$ is written on a piece of paper. June then repeatedly erases every 7th digit of the number. When he runs out of digits to erase, he begins this erasing process over again and again, starting from the beginning of the remaining number. At the end of this process, a six-digit number remains. What is this number?

13. Consider the expression A X B Y C, where (A,B,C) is some arrangement of (0,1,2) and (X,Y) is some arrangement of $(+,\times)$. (For example, it could represent $2+0\times1$.) How many different values, respecting order of operations, can the expression A X B Y C take?

14. A circle is inscribed in a square, which is inscribed in a circle, which is inscribed in a square. The smaller circle has radius 1. Color red the regions inside the larger circle that are outside the smaller square, and color red as well the interior of the smaller circle. In terms of π, what is the total area that is colored red?

15. What is the largest n such that $1^1+2^2+\cdots+n^n\le 50000$?

16. Circles O_1, O_2, and O_3 are mutually externally tangent and have radii of 1, 2, and 3 respectively. What is $[O_1O_2O_3]$?

17. If $1\le x\le 2015$ and $5\le y\le 403$, what is the least possible value of $\frac{x+y}{xy}$?

18. Compute the greatest prime factor of $6^{12}+12^6$.

19. Ryan the $\pitchfork$ randomly draws two points on the circumference of a circle. What is the probability that the points lie within 60° of one another?

20. The graphs of $y=\frac{1}{x}, y=1, y=x$ and $y=x^2$ cut the plane into how many pieces?

21. Alex the Kat is going on a road trip after his retirement. He travels the first 300 miles at 60 miles per hour, the next 90 miles at 90 miles per hour, and the last 360 miles at 40 miles per hour. What is his average speed throughout the trip?

22. Evaluate $1 + (1 + 2) + (1 + 2 + 3) + \cdots + (1 + 2 + 3 + ... + 25)$.

23. If the sum of the digits of a 3-digit number is 15, and the number itself is a multiple of 35, what is this number?

24. If an ant starts at the point $(0, 1)$ and takes a step of length 1 perpendicular to the line between itself and the origin every second, how far from the origin will the ant be after 2015 seconds?

25. Three faces of a rectangular prism have areas 16, 20, and 45. What is the volume of this prism?

26. A coin lands on heads with likelihood $\frac{2}{3}$ and tails with likelihood $\frac{1}{3}$. If Ryan the $\pitchfork$ tosses the coin 4 times, what is the probability that Ryan the $\pitchfork$ will see at least two heads?

27. Abhi, Boris, and Sandy are working all together to shovel a yard full of snow. Abhi and Boris, together, can shovel the yard full of snow in 3 hours. Abhi and Sandy, together, can shovel the yard full of snow in 4 hours. Boris and Sandy, together, can shovel the yard full of snow in 5 hours. How many hours will it take Abhi, Boris, and Sandy, if they all work together, to shovel the yard full of snow?

28. The real numbers $a - 7$, $b + 4$, and $c + 6$ form a geometric sequence, and a, b and c lie in an arithmetic sequence and have an arithmetic mean of 46. What is the arithmetic mean of the terms in the geometric sequence?

29. Each of Jon, Alex, Mike, Soonho and Claire is either a Liar or a Truth-teller.

    - Jon says, "There are no Liars here."
    - Alex says, "There is at least one Liar here."
    - Mike says, "Jon and Claire are both Liars."
    - Soonho says, "There are no more than 4 Liars."
    - Claire says, "Jon and Mike are both Liars."

    How many Liars are there among the five?

30. Find the sum of the x-intercepts of the graph of $y = x^2 - 2015|x| + 20152015$.

31. What is the remainder when $1^1 + 2^2 + \cdots + 7^7$ is divided by 8?

32. Hypotenuse AB of right $\triangle ABC$ has length 6. Let H_1 and H_2 be regular hexagons with side lengths AC and BC, respectively. If $[H_1] + [H_2]$ can be written in the form $a\sqrt{b}$, when b is not divisible by any perfect square other than 1, what is $a + b$?

33. Young Guy seems to have caught himself in quite a predicament! Up at the board in his math class, he has been asked by his teacher, Dr. Lal, to compute the product of two numbers a and b, not necessarily distinct, each between 1 and 9, inclusive. But alas, Young Guy has forgotten which two numbers he was assigned, and decides to choose a random pair of numbers and carry out the multiplication. What is the sum of all possible products $a \times b$ Young Guy can reach?

34. The circles O_1 and O_2 have centers that are 9 apart and have radii 4 and 7 respectively. The circles intersect at A and B. Find $[O_1AO_2B]$.

35. Zi Xuan (David) Ni begins at BCA, picks a random direction, and walks in a straight line at a constant pace for 5 minutes. After 5 minutes, he chooses another random direction, and again walks in a straight line for 5 minutes at the same pace as before. What is the probability that, after 10 minutes, he is further from BCA than he was after 5 minutes?

36. $a_0, a_1, ...a_k$ are positive integers that sum to 2014. What is the greatest integer n such that 2^n divides the maximum value of $a_0 \times a_1 \times \cdots \times a_k$?

37. If $x^2 - 13x + 1 = 0$, what is the unit digit of $x^4 + x^{-4}$?

38. AJ the Dennis rolls 3 fair 6-sided dice. He notices that his largest number is exactly twice his smallest number. What is the probability that this happens?

39. Rebecca can only move parallel to the x-axis (at a constant speed of 20 mph) and parallel to the y-axis (at a constant speed of 10 mph), while The Great Bustard can move in any direction at 10 mph. If both Rebecca and The Great Bustard start at the origin, what is the area of the region $|x| < 1$ and $|y| < 1$ in the xy plane that The Great Bustard can reach before Rebecca?

40. A bug on a number line labeled with points $-10, -9, \ldots, 9, 10$ moves one unit left or right every second. If it starts at point 0, what is the probability that it will be at point 1 in nine seconds?

41. Compute the infinite sum $\frac{4}{3} + \frac{7}{9} + \frac{10}{27} + \frac{13}{81} + \cdots$

42. Matthew the Robot is at the top left corner of a 6×6 unit square. He must to walk to the bottom right corner of the board, but has only been programmed to take unit steps right or down. However, the center 2×2 square was cut out, so Matthew cannot walk inside that square (though he can walk along its sides). How many ways are there for Matthew to walk to the bottom right corner?

43. Jen is looking at 12-hour analog clock whose hour hand and minute hand are of lengths 2 and 3 respectively. If Jen looks at the clock at a random time between 12 PM and 1 PM, what is the probability that the distance between the tips of the hour and minute hands is less than or equal to $\sqrt{7}$?

44. If a and b are positive real numbers that satisfy $\frac{1}{a} - \frac{1}{b} - \frac{1}{a+b} = 0$, what is the value of $(a/b)^2 + (b/a)^2$?

45. Suppose N is the number of perfect squares that divide 2015^{2015}. How many perfect squares divide N?

46. The 2 integer roots of the equation $x^2 + cx + a = 0$ are both 1 larger than the roots to the equation $x^2 + ax + b = 0$. Find all possible values of $a + b + c$.

47. Real numbers a, b, and c satisfy these conditions: $a + b + c = 1$, and

$$\frac{1}{a+b-c} + \frac{1}{b+c-a} + \frac{1}{c+a-b} = 1.$$

What is the value of abc?

48. Let $f(x) = \frac{x^2+4x+13}{3x^2+2x+3}$, M be the maximum of f over all reals, and m be the minimum of f over all reals. Find the product of M and m.

49. 27 boxes are used to make a $3 \times 3 \times 3$ cube, with a magical ball that starts in the center box. Each second, the ball randomly moves to a box adjacent to its box at that moment, going in any of the available options with equal likelihood. After some amount of time, what is the probability that the magic ball is in the middle box?

50. A $162 \times 98 \times 63$ prism is built out of unit cubes. If a diagonal is drawn through the prism, how many unit cubes contain positive length of diagonal?