Joe Holbrook Memorial Math Competition

5th Grade Solutions

October 9th, 2016

1. Since half an hour is 30 minutes, Kelvin the Frog can listen to his favorite song $\frac{30}{6} = \boxed{5}$ times.

2. The list of numbers consists of integers between -17 and 17, which includes 0. Any product of a number and 0 is $\boxed{0}$.

3. $$2 + (0 - (1 \cdot 6(2^{0 \cdot \frac{1}{6}}))) = 2 + (0 - (1 \cdot 6(2^0))) = 2 + (0 - (1 \cdot 6)) = 2 - 6 = \boxed{-4}.$$

4. Alex the Kat needs to write x questions such that $25 + x = 40$, so $x = 40 - 25 = \boxed{15}$.

5. A total of 5+2=$\boxed{7}$ points were scored.

6. A hexagon has 6 sides, and a triangle has 3 sides, so a hexagon has $6 - 3 = \boxed{3}$ more sides than a triangle does.

7. Three of the twelve months start with a J: January, June, July. Therefore, $\frac{3}{12} = \boxed{\frac{1}{4}}$.

8. $2, 4$, and 6 are the even numbers on a regular six-sided die. Because each number has an equal likelihood of being rolled, the probability of rolling an even number is $\frac{3}{6} = \boxed{\frac{1}{2}}$.

9. The total tests graded can be calculated by mutiplying 3 and 150, which yields $\boxed{450}$.

10. There are twelve inches per foot, so Yousun is $5 * 12 = 60$ inches tall. Youjung is therefore $60 + 6 = \boxed{66}$ inches tall.

11. The phone has a maximum battery life of $10 + 60 = 600$ minutes. Therefore, the phone has $12\% * 600 = \boxed{72}$ minutes left.

12. By the formula that says the interior angle of a regular n-sided polygon can be found by $\frac{180 \cdot (n-2)}{n}$, we can see that an 8-sided polygon (octagon) has angles of 135 degrees and a 5-sided polygon (pentagon) has angles of 108 degrees. Therefore, 8+5=$\boxed{13}$.

13. The prime factorization of 2016 is $2^5 \cdot 3^2 \cdot 7$. The prime factorization of 2772 is $2^2 \cdot 3^2 \cdot 7 \cdot 11$. The greatest common factor can be found by identifying the least exponent of each prime factor: $2^2 \cdot 3^2 \cdot 7 = \boxed{252}$.

14. Aligning our multiplicands in vertical fashion, we see that many numerators and denominators cancel, leaving a final answer of $\boxed{\frac{1}{6}}$.

15. $\frac{5}{55} = \frac{1}{11}$. $111 \cdot 5555 = \frac{5555}{11} = \boxed{505}$.

16. $2^11 = 2048$, and $2^10 = 1024$ Therefore, the smallest n satisfying the equation is $\boxed{11}$.

17. Four years pass between Kelvin's 4th and 8th grades, which means that his scores improved 4 times. During this period, his score increased by $43 - 31 = 12$ points. Since he increased by an equal amount every year, we divide the total increase by the total time to get $\frac{12}{4} = 3$. This means that in 7th grade he scored 3 fewer points than in 8th grade, or $43 - 40 = \boxed{40}$.

18. Let's call our number n. We then perform many operations: first we obtain $n + 2016$, then $4 \cdot (n + 2016)$, then $4 \cdot (n + 2016) - 12$, then $\frac{4 \cdot (n + 2016) - 12}{4} = n + 2016 - 3 = n + 2013$, then $(n + 2013) - n = \boxed{2013}$.

19. Each of the fractions here are equal to $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \boxed{\frac{1}{16}}$.

20. Since doubling any number gives us an even number, we have to work backwards. On Friday, Kelvin had $\frac{48}{2} = 24$ lilypads. On Thursday, he had $\frac{24}{2} = 12$ lilypads. On Wednesday, he had $\frac{12}{2} = 6$ lilypads. On Tuesday, he had $\frac{6}{2} = 3$ lilypads, making our day $\boxed{\text{Tuesday}}$.

21. The sum of integers from 1 through n can be found by the formula $\frac{n(n+1)}{2}$. Plugging in $n = 63$ results in $\boxed{2016}$. For more information on this fascinating topic, look up "triangle numbers" – you'll encounter them often in the future!

22. Two out of every three cakes were thrown out, which means that one in every three cakes was kept. This is $\frac{1}{3}$ of 132, which is $\boxed{44}$.

23. Let a slice of plain pizza cost x dollars and a slice of pepperoni pizza cost $x + 0.5$ dollars. David and June ordered 3+2= 5 slices of plain pizza and $2 + 4 = 6$ slices of pepperoni pizza, so $5x + 6(x + 0.5) = 11x + 3 = 25$. Since $x = 2$ from the previous equation, a slice of plain pizza costs \$2 and a slice of pepperoni pizza costs \$2.50. David ordered 3 slices of plain pizza and 2 slices of pepperoni pizza, so he paid $3 \cdot 2 + 2 \cdot 2.5 = 6 + 5 = \boxed{11}$ dollars.

24. There are 60 seconds in a minute, so his song is $3 \cdot 60 + 45 = 225$ seconds long. There are 60 minutes in an hour, so there are $60 \cdot 60 = 3600$ seconds in an hour. Thus the final fraction is $\frac{225}{3600} = \boxed{\frac{1}{16}}$.

25. There are 3 different choices for buying milk, 2 different choices for buying eggs, and 4 different choices for buying butter, so the total number of ways to buy one of each is equal to $3 * 2 * 4 = \boxed{24}$.

26. Using the formula for average , we have $\frac{34 + 35 + 39}{3} - \frac{23 - 14 - 17}{3} = \frac{34 + 35 + 39 - 23 - 14 - 17}{3} = \boxed{18}$.

27. Using Vieta's formula, the sum is $\frac{-(-2)}{1} = \boxed{2}$.

28. 54 flips $= 18 \cdot 5 = 90$ flops. 90 flops $= 10 \cdot 14 = \boxed{140}$ flaps.

29. There is a probability of $\frac{1}{3}$ of pulling out the letter B first; then a probability of $\frac{1}{2}$ of pulling out C; the letter A then has a $\frac{1}{1}$ chance of being selected. Multiplying the fractions together gives a total probability of $\boxed{\frac{1}{6}}$.

30. Arthur ran 40 meters in the first 5 seconds. He only has to run for $\frac{100 - 40}{3} = \frac{60}{3} = 20$ more seconds. Sunny ran for 32 meters in the first 8 seconds. That means that in $25 - 8 = 17$ seconds, he must run 68 meters, which is an average speed of $\frac{68}{17} = \boxed{4}$ m/s.

31. The number of permutations disregarding the repeated alphabat is $5! = 120$. However, the letter M is repeated twice, thus the number should be divided by 2!, yielding $\boxed{60}$ as the answer.

32. Note that 2^4 has a units digit of 6. Since $2^{2016} = (2^4)^{504} = 6^{504}$, and every power of 6 ends in 6, we know 2^{2016} has a units digit of 6. Also note that $3^4 = 81$ has a units digit of 1. Since $3^{2016} = (3^4)^{504} = 81^{504}$, we know 3^{2016} has a units digit of 1. Our answer is therefore $1 + 6 = \boxed{7}$.

33. Recall that $2015 = 5 \cdot 13 \cdot 31$, $2016 = 2^5 \cdot 3^2 \cdot 7$, and 2017 is prime. The number of factors of a positive integer n with prime factorization $p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ is $(e_1 + 1)(e_2 + 1) \cdots (e_k + 1)$. Thus, $A = 2 \cdot 2 \cdot 2 = 8$, $B = 6 \cdot 3 \cdot 2 = 36$, and $C = 2$, with average value $\frac{A + B + C}{3} = \boxed{\frac{46}{3}}$.

34. The resulting figure is a semicircle with radius $\sqrt{2}$, and two half-squares with side length 1, which yields an area of $\boxed{\pi + 1}$.

35. The ratio of the areas is $\dfrac{360}{40} = 9$, hence the ratio of the sides will be $\sqrt{9} = 3$. The length h of the larger hypotenuse will satisfy $\dfrac{h}{15} = 3$, and we find $h = \boxed{45}$.

36. Since $\dfrac{x}{x+2} < \dfrac{61}{64}$, multiplying both sides of the inequality by $64(x+2)$ yields $64x < 61(x+2)$, which can be simplified to $3x < 122$, then the largest integer value for x would be $\boxed{40}$.

37. Consider Hannah and Julia as if they were joint together into one block "person", so now there are 6 people. There are $6! = 12$ ways to arrange them in a line. Now we "unravel" the block: either Hannah is to the left of Julia, or Julia is to the left of Hannah. Thus we double the number for a total of $2 \cdot 6! = \boxed{1440}$ different ways to arrange them.

38. We write $3.\overline{703} = 3.703703703\ldots$ as $10 \cdot 0.\overline{370} = 10 \cdot 0.370370370\ldots$, so our fractional expression will be equivalent to $10 \cdot \dfrac{370}{999}$. Since $999 = 27 \cdot 37$, we have $\dfrac{10 \cdot 370}{999} = \dfrac{10 \cdot 10 \cdot 37}{27 \cdot 37} = \boxed{\dfrac{100}{27}}$.

39. We can subtract equation 1 from equation 2 to get that $(3a^2 + 5b^2 + 7c^2 + 9d^2 + 11e^2 + 13f^2) - (a^2 + 3b^2 + 5c^2 + 7d^2 + 9e^2 + 11f^2) = 2(a^2 + b^2 + c^2 + d^2 + e^2 + f^2) = 40 - 20 = 20$. Then adding that with equation 2, $(3a^2+5b^2+7c^2+9d^2+11e^2+13f^2)+2(a^2+b^2+c^2+d^2+e^2+f^2) = 5a^2+7b^2+9c^2+11d^2+13e^2+15f^2 = 40+20 = \boxed{60}$.

40. We can see that the quadrilateral $XABY$ is a trapezoid, and since XY is tangent to circles A and B, $\angle XYB$ and $\angle YXA$ are both right. Since the sum of the angles in a trapezoid is 360 degrees, $\angle XAB + \angle YBA = 180$. We can see that $\triangle XAC$ and $\triangle YBC$ are isosceles, as two of their sides are radii. If we let $\angle XAC = \angle XAB = \alpha$, then $\angle YBC = \angle YBA = 180 - \alpha$. Next, $\angle ACX = \dfrac{180 - \alpha}{2}$, and $\angle BCY = \dfrac{180 - (180 - \alpha)}{2} = \dfrac{\alpha}{2}$. Since $\angle XCY = 180 - (\angle ACX + \angle BCY)$, $\angle XCY = 180 - 90 = \boxed{90}$ degrees.

41. There are 2 cases in which both balls are the same color: Either both are green, or both are blue. P(both are green) $= \dfrac{10}{16} * \dfrac{8}{N+8}$

P(both are blue) $= \dfrac{6}{16} * \dfrac{N}{N+8}$

The sum of these probabilities must be $0.575 = \dfrac{23}{40}$

$\dfrac{6N + 80}{16 \cdot (N+8)} = 23/40$. Cross-multiplying, we get: $240N + 3200 = 368N + 2944$ $256 = 128N \rightarrow N = \boxed{2}$.

42. By difference of squares, the expression becomes $\dfrac{(5^{2016} + 5^{2014})(5^{2016} - 5^{2014})}{(5^{2015} + 5^{2013})(5^{2015} - 5^{2013})}$. After factoring out 5^{2014} from the numerator and 5^{2013} from the denominator: $\dfrac{5^{2014} \cdot (5^2+1) \cdot 5^{2014} \cdot (5^2-1)}{5^{2013} \cdot (5^2+1) \cdot 5^{2013} \cdot (5^2-1)}$. After cancellation, the result is $5 \cdot 5 = \boxed{25}$.

43. Let $x = 2016$. Therefore, we get $\sqrt{(x)(x+1)(x+2)(x+3)+1}$. By multiplying by pair, we get $\sqrt{(x^2+3x)(x^2+3x+2)+1}$, which we can multiply out to get $\sqrt{(x^2+3x)^2 + 2(x^2+3x)+1}$, which then factors out to $x^2 + 3x + 1$. Then, you just plug back in $x = 2016$ and solve to get $\boxed{4070305}$.

44. Let $z = x + 5$. Thus the equation becomes

$$(z-3)(z-1)(z+1)(z+3) = (z-3)^2 + (z-1)^2 + (z+1)^2 + (z+3)^2 + 4$$

This evaluates to,

$$(z^2 - 9)(z^2 - 1) = 4z^2 + 24$$

Which is also,

$$z^4 - 10z^2 + 9 = 4z^2 + 24$$

If this is solved as a quadratic in z^2 and then the solutions for z are substituted back in to get the values of x, it can be seen that the only real solutions for x are $\boxed{-5 \pm \sqrt{15}}$.

45. Let

$$x = \sqrt{7+\sqrt{13}} - \sqrt{7-\sqrt{13}}$$

Then,

$$x^2 = 7+\sqrt{13} - 2\sqrt{49-13} + 7 - \sqrt{13} = 14 - 12 = 2 \Rightarrow x = \sqrt{2}$$

We can quickly check that $x \neq -\sqrt{2}$ by noticing that $\sqrt{7+\sqrt{13}} > \sqrt{7-\sqrt{13}}$. However, the problem statement asks for the answer in the form $a\sqrt{b}$. We have $a = 1$ and $b = 2$, so $a + b = \boxed{3}$.

46. Let $Q(x) = P(x-2)$. Then, $Q(x) = (x-2)^1 0 + 2(x-2)^9 + 4(x-2)^8 + 8(x-2)^7 + R(x)$, where $R(x)$ is a polynomial of degree 6. The coefficient of the term with degree 7 is therefore $(-1 \cdot \binom{10}{3} \cdot 2^3) + (2 \cdot \binom{9}{2} \cdot 2^2) + (-4 \cdot \binom{8}{1} \cdot 2^1) + (8 \cdot \binom{7}{0} \cdot 2^0) = -960 + 288 - 64 + 8 = \boxed{-728}$.

47. Our new polynomial is of the form $x^3 + bx^2 + cx + d$. By Vieta's, $-b = pq + pr + qr$, which equals -1 from the original polynomial. Likewise, $-d = pq \cdot pr \cdot qr = p^2q^2r^2 = (-30)^2 = 900$. Finally, $c = pq \cdot pr + pq \cdot qr + pr \cdot qr = p^2qr + pq^2r + pqr^2 = pqr(p+q+r) = (-30)(6) = -180$. Now, we can plug in the values we found for each coefficient into the polynomial, which yields $x^3 + x^2 - 180x - 900$. Thus, our answer is $1 - 180 - 900 = \boxed{-1079}$.

48. Let $S = \frac{1}{2} + \frac{3}{4} + \frac{5}{8} + \frac{7}{16} + \ldots$. Then we know that $\frac{1}{2}S = \frac{1}{4} + \frac{3}{8} + \frac{5}{16} + \ldots$. Subtracting yields that

$$\frac{1}{2}S = \frac{1}{2} + \frac{2}{4} + \frac{2}{8} + \frac{2}{16} + \ldots$$

Hence we have,

$$\frac{1}{2}S = \frac{1}{2} + \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\right)$$

The expression in the parenthesis is a geometric series with starting term $\frac{1}{2}$ as well as a common ratio of $\frac{1}{2}$ Thus the expression in the parentheses evaluates to $\frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$. Multiplying the overall equation by 2, we have $S = 2 \cdot (\frac{1}{2} + 1) = \boxed{3}$.

49. Since $2016^2 = 2^{10} \cdot 3^4 \cdot 7^2$, so $k = (10+1) \cdot (4+1) \cdot (2+1) = 165$. Notice for every d_i other than 2016, there is a d_j such that the product of d_i and d_j is 2016^2. Now consider the sum $\frac{1}{d_i + 2016} + \frac{1}{d_j + 2016}$:

$$\begin{aligned}\frac{1}{d_i+2016} + \frac{1}{d_j+2016} &= \frac{1}{d_i+2016} + \frac{1}{\frac{2016^2}{d_i} + 2016} \\ &= \frac{1}{d_i+2016} + \frac{d_i}{2016d_i + 2016^2} \\ &= \frac{2016}{2016d_i + 2016^2} + \frac{d_i}{2016d_i + 2016^2} \\ &= \frac{1}{2016}\end{aligned}$$

Among the 165 divisors of 2016^2 there are $\frac{165-1}{2} = 82$ pairs of such d_i and d_j. Therefore, the desired sum is $82 \cdot \frac{1}{2016} + \frac{1}{2016+2016} = \boxed{\frac{165}{4032}}$.

50. Let Q be the midpoint of BC. Since $\triangle BMQ \sim \triangle BAC$ with a $1:2$ ratio, $QM = AC/2 = 6$. Similarly, $QN = BD/2 = 9$. Then, from right triangle MQN, we have that $MN = \sqrt{9^2 + 6^2} = \sqrt{117}$. Finally, from right triangle MNP, $MP = \sqrt{117 - 36} = \boxed{9}$.