Joe Holbrook Memorial Math Competition

6th Grade Solutions

October 15, 2017

1. It is 2 hours and 36 minutes from 3:00, and as there are 60 minutes in an hour, $2 \cdot 0 + 36 = \boxed{156}$.
2. Note that there are 60 minutes in an hour and 60 seconds in a minute, so traveling 60 miles would take 60 minutes and $7 \cdot 60 = 420$ seconds, or $60 + 7 = \boxed{67}$ seconds.
3. Since addition and subtraction come before division and mutiplication, $2 + 5 \cdot 2 - 5 = (2 + 5) \cdot (2 - 5)$. Now, since operations are performed right to left, $(2 + 5) \cdot (2 - 5) = 7 \cdot 3 = \boxed{21}$.
4. The larger square has side lengths of 6, which means that it has a perimeter of 24. Meanwhile, the smaller square, which has side lengths 3, has perimeter 12. Therefore, their collective perimeter is 36. Since the side of the smaller square is glued to the side of the larger square, you must subtract the edge between them, which is 3 for each square. Therefore, $36 - 6 = \boxed{30}$.
5. $20\% \cdot 300 = 60$ of the donuts had sprinkles. Likewise, 30 had chocolate, and 111 had jelly. Thus, there were $300 - 60 - 30 - 111 = \boxed{99}$ plain donuts.
6. It takes Jaylen $\frac{2}{3}$ hours, or 40 minutes, to get to the park, while it only takes Derrick $\frac{2}{5}$ hours, or 24 minutes. Consequently, Jaylen must leave his house $40 - 24 = \boxed{16}$ minutes earlier than Derrick does.
7. The problem is about finding the LCM of 9 and 60, which is 180. That is 3 hours later, which is $\boxed{\text{3:00 PM}}$
8. Call the original number of boats x. The condition tells us that $6(x + 1) = 9(x - 1)$, which can be solved and $x = 5$. Thus the number of people is $6 \cdot (5 + 1) = \boxed{36}$.
9. We have to choose the remaining 2 members out of the 5 other people, so the answer is $\binom{5}{2} = \boxed{10}$.
10. Note that $2 < \sqrt{7} < 3$ and $26 < \sqrt{700} < 27$, so whole numbers that satisfy this condition are $3, 4, \ldots, 26$. Therefore, the answer is 26 - 3 + 1 = $\boxed{24}$.
11. $12 \cdot 1.1 \cdot 0.9 = \boxed{11.88}$. Note that order does not matter in performing this calculation!
12. He scored a total of $25 \cdot 7 = 175$ points over all 7 games. In the first 4, he scored a total of $4 \cdot 14 = 56$ points, while in the last four games, he scored a total of $4 \cdot 31 = 124$ points. Adding these together, we get $124 + 56 = 180$, but we overcounted the points he scored in the 4th game once, so the answer is the difference: $180 - 175 = \boxed{5}$ points in his 4th game.
13. Note that the units digit of 7^n repeats in cycles of 4. Since $\frac{2017}{4} = 504 + \frac{1}{4}$, the unit digit is the same as $7^1 = \boxed{7}$.
14. The given inequality can be factored as $(n - 1)(n - 2) \leq 0$. For this to hold, n must satisfy $1 \leq n \leq 2$. There are only two integers in this range, namely $\boxed{1, 2}$.
15. To find how many "good" sandwiches there are, we need to subtract the number of "bad" sandwiches from the total number of possible sandwiches. There are 4 sandwiches with salami and american cheese, and 3 sandwiches with roast beef and baloney, resulting in 7 "bad" sandwiches. The total number of sandwiches is 30 ($\binom{5}{2} \cdot \binom{3}{1} = 10 \cdot 3$). Subtracting $30 - 7 = \boxed{23}$ good sandwiches.
16. Let x be the number of times that David Ni has hugged a dog and y be the number of times that he has hugged a cat. Since he spends \$2 to hug a dog and \$10 to hug a cat, $2x + 10y = 200$. He hugged a total of 32 times, so $x + y = 32$. Then, solving $2x + 10y = 200$ and $x + y = 32$ gives $x = 15, y = 17$ We are looking for the number of times David hugged a cat so the answer is $\boxed{17}$.

17. Note that $0\Delta a = a\Delta 0 = 0$ if $a \neq 0$. Thus $((2\Delta 6)\Delta 3)\Delta(0\Delta(3\Delta 7)) = \boxed{0}$.

18. Since the sequence starts off with 2 odd terms, every third term will be even. Thus, all the terms that are multiples of three are the ones we are looking for. Among the first 100, thee are $\boxed{33}$ multiples of 3.

19. In those 5 numbers, there must be at least 2 5's in order to make it the unique mode. In order to minimize the average of the smallest 4 numbers, you want 1 and 2 to be part of the set (you can't have 2 1's since that would mean that 5 is not the unique mode). Now, given these 4 numbers, you need $\boxed{12}$ in order to make the average 5.

20. 0 cannot be a base number since that makes the whole expression 0. Then, it must be an exponent number to one of $\{2, 1, 7\}$. Since 1 is the lowest of the three and has the least impact, we choose that one. Since $2^7 > 7^2$, our end expression is $2^7 \cdot 1^0 = \boxed{128}$.

21. Triangles $\triangle ABD$ and $\triangle BDC$ are equilateral since $\frac{120°}{2} = 60°$ so the total area is $2 * \frac{6^2\sqrt{3}}{4} = \boxed{18\sqrt{3}}$

22. Looking at the hundreds place, the two digits have to be 9 and 1. Now, if the tenth place of the subtrahend is 1, the smallest it can be is 110, and the minuend would have to be more than 894+110=1004. Thus the tens place of the subtrahend is $\boxed{0}$, and so is the product of the digits.

23. Since Haneul and Ivy always sit together, we can consider them as one person. Once we have done this, the only "person" Julia can sit next to is the pair Haneul and Ivy. Therefore Julia must be on the end, and she must be next to Haneul and Ivy. This pair can be oriented two ways, and Julia can be on the left or the right, and Ben and David can be oriented two ways, so there are $\boxed{8}$ total permutations.

24. Being a multiple of 3 depends on the sum of the digits of each term. The term has remainder 0 divided by 3, the second term has remainder 0, the third term has remainder 2, and the last term has remainder 1, so the sum has remainder 0 divided by 3. Therefore, it is always a multiple of 3 regardless of the position of the digits in each term so the probability is $\boxed{1}$.

25. By properties of $30-60-90$ degree triangles, the height of the tree must be $6\sqrt{3}$ feet tall. As the triangle formed by the position of the bug, David, and the base of the tree forms another $30 - 60 - 90$ degree triangle, the height of the bug above the ground is $\frac{6}{\sqrt{3}} = \boxed{2\sqrt{3}}$ feet.

26. Chord BP is a diameter of Ω as point O lies on BP. As the radius of Ω is 2, the length of BP is 4, and the angle $\angle BCP = 90°$, applying the Pythagorean Theorem on $\triangle BCP$ yields $BC^2 + PC^2 = BP^2$. Plugging in the given values yields $BC^2 = 4^2 - 3^2 = 7 \rightarrow BC = \boxed{\sqrt{7}}$.

27. $1200 = 2^4 \cdot 3 \cdot 5^2$, so there are $5 \cdot 2 \cdot 3 = 30$ total factors and $3 \cdot 1 \cdot 2 = 6$ perfect square factors. Therefore, there are $\boxed{24}$ non-perfect square factors of 1200.

28. As $AB = 8$ and $BC = AD = 6$, by the Pythagorean Theorem, it follows that $BD = 10$. As M is the midpoint of BD, DM must be 5. As AN is the height to the base BD in triangle $\triangle ABD$, $\frac{1}{2} \cdot AN \cdot BD = \frac{1}{2} \cdot AB \cdot AD \rightarrow AN = \frac{24}{5}$. By the Pythagorean Theorem in triangle $\triangle AND$, $DN^2 = AD^2 - AN^2 \rightarrow DN = \frac{18}{5} \rightarrow MN = \boxed{\frac{7}{5}}$.

29. Set side BC to be of length x. Then side CD is of length $37.5 - x$, since the sum of the two sides is half the perimeter. Then, $x \cdot 14 = (37.5 - x) \cdot 16$, solving gives us $x = 20$. Thus the area is $20 \cdot 14 = \boxed{280}$

30. The horse can graze in $360° - 60° = 300°$ of a circle radius 8 which equates to $\frac{300}{360} * \pi * 8^2$ area. It can also graze in the two segments touching the opposite edge of the triangle to the vertex it is tied to, each of which is $180° - 60° = 120°$ of a circle radius $8 - 6 = 2$, equating to $2 * \frac{120}{360} * \pi * 2^2$ area. So the final answer is $\boxed{56\pi}$.

31. There are 5 people who are kicked out and 5 who stay. Since the likelihood of being in either of those groups is the same, the answer is $\boxed{1/2}$.

32. Each human does 25%, a dolphin 12.5%, a chicken 2.5%, and a rat 2% of the job of screwing in a light bulb in 1 hour and when the time is halved to 30 minutes, each animal does half of the job in that time. Since there is a human, 5 dolphins, and 25 chickens, they do $(12.5 \cdot 1 + 6.25 \cdot 5 + 25 \cdot 1.25)\% = 75\%$ of the job in 30 minutes. Since each rat does 1% of the job in 30 minutes, we need $100 - 75 = \boxed{25}$ rats.

33. The condition is $2a^2 + b = b^2 + ab + b$, which simplifies to $(a - b)(2a + b) = 0$. Since a and b are different, $2a + b = 0$. $f(2)$ is $4 + 2a + b$, so the desired value is $\boxed{4}$.

34. The system:

$$n \equiv 5 (\text{mod } 7)$$
$$n \equiv 4 (\text{mod } 9)$$

has solutions $n \equiv 40 (\text{mod } 63)$, so the third smallest solution is $40 + 2 * 63 = \boxed{166}$.

35. There are a total of $\binom{12}{5} = 792$ routes if the restrictions are not considered. Use complementary counting: There are $\binom{4}{1} \cdot \binom{8}{4} = 280$ routes that go through $(1, 3)$, and $\binom{4}{1} \cdot \binom{8}{2} = 112$ routes that go through $(3, 1)$. Therefore, there are $792 - 280 - 112 = \boxed{400}$ routes from $(0, 0$ to $(5, 7)$ that pass through neither $(1, 3)$ nor $(3, 1)$.

36. Squaring both sides, we get $x^2 + y^2 + z^2 + 2xy + 2yz + 2xz = 2xy + 2yz + 2xz + 41$. The $2xy + 2yz + 2xz$ terms cancel, and the resulting equation is $x^2 + y^2 + z^2 = 41$. There are 9 solutions where $x, y, z \neq 0$: 6 permutations of $1 + 4 + 36$ and 3 permutations of $9 + 16 + 16$, and 6 solutions with exactly one of x, y, z equal to 0: $16 + 25$. Therefore there are $\boxed{15}$ total solutions.

37. As $x^2 + x + 1 = 0$, $x^3 = 1$. Thus, $x^{200} = (x^{66})^3 \cdot x^2 = x^2$ and $x^{100} = (x^{33})^3 \cdot x = x$, so $x^{200} + x^{100} + 1 = x^2 + x + 1 = \boxed{0}$.

38. Let $AB = x$ and $CD = y$. We want to find the length of $MN = \frac{x+y}{2}$. Consider the line l through B parallel to AC. Let l intersect CD at E. Then, $BE = AC = 11$ and $CE = AB = x$. Since $AC \perp BD$ and $AC \parallel BE$, we also have $BE \perp BD$. Thus, by Pythagorean theorem on right triangle $\triangle EBD$, we have that $(x + y)^2 = 60^2 + 11^2 = 3600 + 121 = 3721 = 61^2$, so $x + y = 61$. Hence, $MN = \frac{x+y}{2} = \boxed{\frac{61}{2}}$.

39. First, you want to find slowest the eastbound train can be going for it to not crash with the northbound one. In order for that to happen, the eastbound train has to hit the tail of the northbound one. The northbound train will be completely clear of the intersection in: $(200 + 25)/50 = 9/2$ hours. This means that the head of the eastbound train has to be there at this time, which leads to the minimum speed being: $300/(9/2) = 200/3$. For the maximum speed, the northbound train has to hit the tail of the eastbound train. The time it will take for the tail of the eastbound train to reach the intersection is $(300 + 50)/s$, where s is the speed. We want this to equal $200/50$, the time it takes for the northbound train to reach the intersection. Therefore, solve, and get that s is $175/2$. Therefore, take their difference, and you get $\boxed{125/6}$.

40. Note that

$$\frac{2022!}{2017!} = 2018 \cdot 2019 \cdot 2020 \cdot 2021 \cdot 2022,$$

so $2020 \mid \frac{2022!}{2017!}$, which implies that $101 \mid \frac{2022!}{2017!}$, so

$$33, 632, 280, 3AB, 168, 080 \equiv 0 \ (\text{mod } 101).$$

Since $100 \equiv -1 \ (\text{mod } 101)$, this implies that $33, 632, 280, 3AB, 168, 080$ is congruent to

$$3(-1)^8 + 36(-1)^7 + 32(-1)^6 + 28(-1)^5 + 3(-1)^4 + (\overline{AB})(-1)^3 + 16(-1)^2 + 80(-1)^1 + 80(-1)^0$$

$$\equiv 3 - 36 + 32 - 28 + 3 - \overline{AB} + 16 - 80 + 80$$

$$\equiv -\overline{AB} - 10 \equiv 0 \ (\text{mod } 101).$$

Thus, $\overline{AB} \equiv -10 \equiv 91 \ (\text{mod } 101)$, and since A and B are digits, this implies that $\overline{AB} = 91$. Hence, $10A + B = \boxed{91}$.