Joe Holbrook Memorial Math Competition

8th Grade

October 15, 2017

General Rules

- You will have **75 minutes** to solve **40 questions**. Your score is the number of correct answers.
- Only answers recorded on the answer sheet will be graded.
- This is an individual test. Anyone caught communicating with another student will be removed from the exam.
- Scores will be posted on the website. Please do not forget your ID number, as that will be the sole means of identification for the scores.
- You may not use the following aids:
  - Calculator or other computing device
  - Compass
  - Protractor
  - Ruler or straightedge

  In addition, you must use the scrap paper supplied by the proctors.

Other Notes

- Write legibly. If the graders cannot read your answer, you will be given no credit for that question.
- Fractions should be written in lowest terms. Please convert all mixed numbers into improper fractions.
- For constants such as e or π, do not approximate your answer: for example, if the answer to a question is 7π, then you should not write 22 or 21.99.
- You do not need to write units in your answers.
- Rationalize all denominators. In addition, numbers within a square root must be squarefree, e.g. $\sqrt{63}$ should be written as $3\sqrt{7}$.
- Ties will be broken by the number of correct responses to questions 31 through 40. Further ties will be broken by the number of correct responses in the last five questions.

1. In Reverse Land, addition and subtraction come before division and multiplication, and operations are performed from right to left. What is $2 + 5 \cdot 2 - 5$ in Reverse Land?

2. It takes Malik 12.21 seconds to run 100 meters. However, he is the greatest distance runner of all time and his speed remains constant no matter how tired he gets. How many seconds does it take Malik to complete a 5 kilometer race? Express your answer is a decimal.

3. Jaylen is trying to improve his basketball shooting percentage at the gym, and he refuses to go home until he has made at least 70% of the shots he has taken. So far, he has taken 15 shots and made only 7 of them. Assuming that from now on, he never misses a shot, what is the minimum number of shots that he has to make consecutively to reach his goal?

4. If you flip 1000 coins, there is an $n\%$ percent chance that at least 500 will land the same way. Find n.

5. John multiplies all the factors of 42 together, and to his surprise, finds that this is 42^n. What is n?

6. If the least common multiple of positive integers a, b is 20 and the greatest common divisor of a, b is 10, find the product $a \cdot b$.

7. The Fibonacci sequence is a sequence of numbers where any term after the second is the sum of the previous two terms. Among the first 100 terms of the sequence, how many are even? The first 8 terms are given here: 1, 1, 2, 3, 5, 8, 13, 21.

8. What is the maximum number of intersections between 17 circles and 17 lines?

9. The average, median, and unique mode of 5 positive numbers is 5. What is the maximum possible value of the largest number in those positive numbers?

10. In the world of Tri, the inhabitants have three feet. Trigo, an inhabitant of Tri, has 6 red socks, 6 blue socks, and 6 green socks. In the early morning, Trigo grabs 3 random socks and puts them on without looking. What is the probability that all three are of the same color?

11. The difference of two 3-digit numbers is 894. Find the product of the digits of the 2 numbers.

12. Square $ABCD$ has sidelength 1. Points M and N are on sides BC and CD, respectively, such that the perimeter of $\triangle CMN$ is 2. Find the degree measure of $\angle MAN$.

13. Lily flips a perfectly circular quarter of diameter 1 inch on her infinite floor. The floor is tiled with squares with side length 2 inches. What is the probability that the quarter does not intersect any of the sides?

14. Triangle ABC has circumcircle Ω with center O and radius 2. If line BO meets Ω again at a point P other than A with $PC = 3$, find the length of side BC.

15. Let $ABCD$ be a rectangle with $AB = 8$ and $BC = 6$. If M is the midpoint of BD and N is the foot of the perpendicular from A to BD, find MN.

16. Parallelogram $ABCD$ has a perimeter of 75. When side BC is the base, the height is 14. When side CD is the base, the height is 16. Find the area of parallelogram $ABCD$.

17. How many positive integers divide at least two of the numbers 84, 120, and 126?

18. Define a *lattice point* to be a point in the coordinate plane whose coordinates are both integers. Bessie selects a lattice point (x, y) whose coordinates satisfy $1 \leq x, y \leq 10$ uniformly and at random, and draws a line connecting the chosen point with the origin. Compute the probability that the slope of this line is a positive integer.

19. Let a and b be two distinct real numbers. If the function $f(x) = x^2 + ax + b$ satisfies $f(a) = f(b)$, find the value of $f(2)$.

20. Badville is a nation filled with bad highways. There are 20 cities, and every pair of cities is connected by a unique highway, all of which need to be renovated. What is the maximum number of highways the government can close such that there is a path through open highways between any two cities?

21. A kite is inscribed in a circle of radius 5. If two of the sides have side length 6, what is the length of either of the other two sides?

22. Compute the largest positive value of the difference between two terms of x, y, z given that $|x - y| + |y - z| + |z - x| = 20$.

23. What is the probability that when 3 edges are chosen from a cube, they are pairwise skew? Two lines are skew if they are not parallel and do not intersect each other.

24. Suppose you have a circle with center O and diameter 15. Point P is outside the circle and A is a point such that PA is a tangent. Extend A through O to get C on the circle and let B be the intersection of PC and the circle. $PB = 16$. Find PC.

25. Haneul has two distinct fair six-sided dice. If she rolls two numbers x and y and computes $\binom{x}{y}$, what is the expected value of $\binom{x}{y}$? ($\binom{n}{k} = 0$ if $k < n$).

26. Cube $ABCD - EFGH$ has edge length 1. Find the distance from BH to EG.

27. Jane flips 12 fair coins. What is the probability that she gets at least as many heads as tails?

28. Consider $x_1 + x_2 + \cdots + x_{100}$, where $x_i \in 0, 1, 2, 3, \ldots, 99$ for all $i = 1, 2, 3, ..., 100$. Find the number of solutions to $100|x_1 + x_2 + ... + x_{100}$.

29. The numbers $0, 0, 1, 1, 2, 2, 7,$ and 7 are arranged in a sequence. An entry in this sequence is called pure if there is no number after it in the sequence that is greater than it (for example, in the sequence $0, 1, 7, 2, 7, 2, 1, 0$, the right 2 in the arrangement is pure). How many sequences are there such that at least one of the 2's is pure?

30. Wayne is told quadratics usually have 2 solutions, so he decides to try the equation $ax^2 + (4a^2 - 2a)x + (3a^3 + 2a^2 - 4a) = 0$. He figures out this doesn't have 2 solutions. Find all possible values of a.

31. If x satisfies $x^2 + x + 1 = 0$, compute $x^{200} + x^{100} + 1$.

32. Circles of radii $8, 8, 9$ and X are all externally tangent to each other. What is X?

33. Bag A contains 5 red balls and 3 blue balls. Bag B contains 3 red balls and 4 blue balls. A bag is chosen at random, and a ball is drawn from the selected bag. Given that the ball is blue, what is the probability the ball was from Bag B?

34. In right triangle ABC, $BC = 24$ and $\angle B = 90°$. A circle of radius 5 intersects sides AB, BC, and CA at P and Q, R and S, and T and U, respectively, such that $PQ = RS = TU = 6$. What is the length of leg AB?

35. In trapezoid $ABCD$, $AB \parallel CD$, $AC = 11, BD = 60$, and $AC \perp BD$. If M and N are the midpoints of AD and BC, respectively, what is the length of segment MN?

36. Given that $\frac{2022!}{2017!} = 33,632,280,3AB,168,080$, where A and B are digits, what is $10A + B$?

37. $P(x)$ is a monic polynomial of degree 4. If $P(1) = 2$, $P(2) = 4$, $P(3) = 6$, and $P(4) = 8$, what is $P(5)$?

38. Two trains are running at a right angle from each other towards the same intersection. The train running north is running at 50 mph, is 200 miles from the intersection, and is 25 miles long. The train running east is 300 miles from the intersection, and is 50 miles long. What is the difference between the minimum and maximum speed that the train running east can run such that the two trains will collide?

39. Haneul has a fair six-sided die, and she rolls the numbers a, b, c and d in that order. Let P be the polynomial of degree at most 3 such that $P(1) = a + b, P(2) = ab, P(3) = c + d$, and $P(4) = cd$. What is the expected value of $P(5)$?

40. Let n be an arbitrary positive integer. Consider the sequence $\{a_i\}_{i=1}^{2^n}$ defined by $a_1 = 1, a_2 = 0$, $a_i = 2^{k-1} + a_{2^k - i + 1}$ where 2^k is the smallest power of 2 greater than or equal to i for $i \geq 3$. Define an *inversion* to be a pair of terms in the sequence a_i, a_j such that $i < j$ and $a_i > a_j$. Compute the number of inversions over all pairs of terms in a_i in terms of n.