Joe Holbrook Memorial Math Competition

5th Grade Solutions

October 14, 2018

1. $80 \text{ problems} \cdot \dfrac{1 \text{ minute}}{5 \text{ problems}} = \boxed{16}$ minutes.

2. We use PEMDAS: $2 \div (0+1) \cdot 8 = 2 \div 1 \cdot 8 = 2 \cdot 8 = \boxed{16}$.

3. There are 7 months with 31 days and 12 months in a year. Therefore, $\boxed{\dfrac{7}{12}}$ of the months have 31 days.

4. Jenn ate $\dfrac{2}{3} \cdot \dfrac{2}{5} = \dfrac{4}{15}$ of the cake. Therefore, Shalin and Jenn together ate $\dfrac{4}{15} + \dfrac{1}{3} = \dfrac{9}{15}$ of the cake. Thus, $1 - \dfrac{9}{15} = \dfrac{6}{15} = \boxed{\dfrac{2}{5}}$ of the cake remained.

5. Working backwards, we multiply 130 by 4 to find the total number of students: $130 \times 4 = \boxed{520}$.

6. Multiply to find the answer of $4 \cdot 10 \cdot 18 = \boxed{720}$ dollars.

7. Addition gives $1009 + 190 + 910 + 9001 = \boxed{11110}$.

8. The number halfway between $\dfrac{1}{2}$ and $\dfrac{1}{4}$ is the average of $\dfrac{1}{2}$ and $\dfrac{1}{4}$, which is $\dfrac{\frac{1}{2} + \frac{1}{4}}{2} = \boxed{\dfrac{3}{8}}$.

9. On day n, the student solves $3n+1$ problems, since on day 1 she solved $3 \times 1 + 1 = 4$ problems and she solves three more problems than she did the day before. On day 12, she solves $3 \times 12 + 1 = \boxed{37}$ problems.

10. Let x be Mikako's favorite positive number. Then,

$$\sqrt{4x - 11} = 3 \implies (\sqrt{4x-11})^2 = 3^2 \implies 4x - 11 = 9 \implies x = 5.$$

Thus, Mikako's favorite positive number is $\boxed{5}$.

11. Each chopped off head results in 2 more heads. Therefore, the answer is $2 \cdot 7 + 20 = \boxed{34}$.

12. Kelvin needs \$100 and already has \$25, so he needs to earn \$75 from his job. Since he earns \$4 an hour, he will need to work at least $\dfrac{75}{4}$ hours. Since Kelvin only works a whole number of hours, round $\dfrac{75}{4}$ up to the nearest whole number. As a result, Kelvin will need to work $\boxed{19}$ more hours in order to afford his chicken nuggets.

13. Since Herb ate $\dfrac{1}{4}$ of a pizza, there is $\dfrac{3}{4}$ of the pizza remaining. Anna eats half of this portion, so half is left, and half of $\dfrac{3}{4}$ is $\boxed{\dfrac{3}{8}}$.

14. The factors of 2018 are 1, 2, 1009, and 2018. Thus, the sum of the divisors of 2018 is $1 + 2 + 1009 + 2018 = 3030$. The prime factorization of 2019 is $3^1 \cdot 673^1$. Thus, 2019 has 4 factors. The answer is $3030 - 4 = \boxed{3026}$.

15. Since 2019 is odd, it cannot be the sum of two odd prime numbers. Therefore, $2019 = 2 + 2017$ is the only possibility, and there is exactly $\boxed{1}$ way.

16. Let a be the number of adult cows and b be the number of babies. Then, we can set up a system of linear equations with the given information. Since Autumn waits for 38 minutes, $2a + 5b = 38$, and since she sees 10 cows, $a + b = 10$. We then use elimination to solve for a and b. We multiply the second equation by 2, which yields $2a + 2b = 20$, and subtract it from the first equation, which yields $3b = 18$, so $b = \boxed{6}$.

17. 1234567 is slightly greater than $1000000 = 10^6$, so its square is also slightly greater than $(10^6)^2 = 10^{12}$, which has $\boxed{13}$ digits (in fact the square of any number between 10^6 and $\sqrt{10} \cdot 10^6 \approx 3160000$ will have 13 digits).

18. Initially, it may appear that repeatedly multiplying by 2 at the outset gives the optimal number of operations. Using this strategy, the sequence is 2, 4, 8, 16, 32, 33, 34, 35. However, this strategy requires the addition of three 1s. Instead, we can add 1 before the last multiplication by 2, yielding 2, 4, 8, 16, 17, 34, 35. Thus, a minimum of $\boxed{6}$ operations are needed.

19. For $n \geq 2$, the probability that domino n is knocked over is $\frac{n}{n+1}$. Furthermore, for $n \geq 2$, domino n cannot be knocked over unless domino $n-1$ is knocked over. Therefore, the answer is $\frac{2}{3} \cdot \frac{3}{4} \cdots \frac{2018}{2019} = \boxed{\frac{2}{2019}}$.

20. The sum of Andrew's scores on the five tests he has taken is $94 \cdot 5 = 470$. In order to receive an average of 95 across all six tests, the sum of his scores on the six tests must be at least $6 \cdot 95 = 570$. Therefore, he must receive a score of (at least) $570 - 470 = \boxed{100}$ on his sixth test.

21. The least common multiple is $100 = 2^2 \cdot 5^2$. Since 4 has no factors of 5, x must be a multiple of $5^2 = 25$. We see that the only multiples of 25 that are factors of 100 are $25, 50, 100$. All of these work, so the sum is $\boxed{175}$.

22. The 48 couples contain a total of $48 \times 2 = 96$ people. The probability a randomly selected person is not in a couple is therefore $\frac{120-96}{120} = \frac{24}{120} = \boxed{\frac{1}{5}}$.

23. The probability that both hands of the clock are visible is $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ (each hand is independent of the other spends equal time on each half), so both are visible for $24 \cdot \frac{1}{4} = 6$ hours in a day. What we are looking for is the opposite of this, or $24 - 6 = \boxed{18}$ hours.

24. By the problem constraints, the numbers not divisible by 3 must form a contiguous block (the number of factors of 3 is nonincreasing). Similarly, the numbers with one factor of 3 and the numbers with two factors of 3 each form contiguous blocks. Finally, each block must be increasing, so we get the answer: $\boxed{(9, 3, 6, 12, 4, 8)}$

25. There are $\binom{8}{4} = 70$ ways to arrange the chocolate and steak, and $2^4 = 16$ ways to choose which pieces of chocolate are stolen by the server. Therefore, there are $70 * 16 = \boxed{1120}$ ways the king and receive the meal.

26. The region that the lamb can wander in is a circle of radius 6 around the origin, with two circular cutouts of radius 6 from each of the two wolves. It can be noted that if you move the cut-out pieces properly, you can form a square with a diagonal of length 12. Therefore, the area of the region is $\boxed{72}$.

27. We know ABH is a right triangle, with hypotenuse 13 and one leg of length 12, so $BH = 5$. Since $BC = 14$, we now know $CH = 9$, and since ACH is also a right triangle, we have $AC = \sqrt{9^2 + 12^2} = \boxed{15}$.

28. First, note that angle EOG is equal to angle DOB. Also, $OE = OG = 3$ and $OD = OB = 5$. By SAS Similarity, triangle EOG is similar to triangle DOB, with ratio $3 : 5$. Therefore, the ratio of their areas is $(3 : 5)^2 = 9 : 25$. Therefore, $[EOG] = \frac{9}{25}[DOB]$. Using the same logic, the ratio $[GOF] = \frac{9}{25}[DOA]$. Now, $\frac{[ABD]}{[EFG]} = \frac{[DOB] + [DOA]}{[EOG] + [GOF]} = \frac{[DOB] + [DOA]}{\frac{9}{25}[DOB] + \frac{9}{25}[DOA]} = \boxed{\frac{25}{9}}$.

29. The smallest palindrome containing a 5 which is greater than 34543 is 35053. (This can be found by generating palindromes in increasing order: start by incrementing the middle digit, then the tens thousands digits, and so on.) If we write this palindrome in the form displayed by the app, we find that the next time at which Eric can take a screenshot is 3:50:53. This is 5 minutes and 10 seconds after 3:45:43, which yields the answer of $\boxed{310}$ seconds.

30. The sum $\frac{p}{q}+\frac{q}{p}$ can be rewritten as $\frac{p^2+q^2}{pq}=\frac{(p+q)^2-2pq}{pq}$. By Vieta's Formulas, $p+q=-5$ and $pq=-17$. Plugging these values into the above fraction yields the answer of

$$\frac{(p+q)^2-2pq}{pq}=\frac{25+34}{-17}=\boxed{-\frac{59}{17}}.$$

31. Call moving up one unit and left one unit a *type I move,* and call moving up one unit and right one unit a *type II move.* Then, Elaine must make 2 type I moves and 3 type II moves. She can make these moves in any order. Out of her 5 total moves, 2 of them must be of type I, so there are a total of $\binom{5}{2}=\boxed{10}$ ways she can move to $(1,5)$.

32. Let the original slice of cheese have size 1. Then the sizes of the three cut-up pieces, in decreasing order of size, are $\frac{1}{(x+1)^2}$, $\frac{x}{(x+1)^2}$, and $\frac{x}{x+1}$. Setting the former equal to the sum of the latter two yields the equation

$$\frac{1}{(x+1)^2}=\frac{x}{(x+1)^2}+\frac{x}{x+1} \implies x^2+2x-1=0 \implies x=\boxed{\sqrt{2}-1},$$

as $x>0$. Alternative Solution: Since the sum of the sizes of the two smaller pieces is equal to the size of the largest piece, it follows that the largest piece has size $\frac{1}{2}$. Therefore, $\frac{1}{(x+1)^2}=\frac{1}{2} \implies x=\sqrt{2}-1$, as above.

33. Let x denote the distance between the Lyp and the origin. By the Generalized Triangle Inequality, $1+2+x\geq 4 \implies x\geq 1$. In addition, the Generalized Triangle Inequality impies that $x\leq 1+2+4=7$. It follows that Lyp can reach any point that is between 1 unit and 7 units away from the origin. Therefore, the set of all points he can reach is the interior of a sphere of radius 7 with a sphere of radius 1 removed from its center. Thus, the volume of the set of all points he can reach is

$$\frac{4\cdot\pi\cdot 7^3}{3}-\frac{4\cdot\pi\cdot 1^3}{3}=\frac{4\cdot\pi\cdot(7^3-1^3)}{3}=\boxed{456\pi}.$$

34. Let points A and B be the centers of circles Γ_1 and Γ_2, respectively. Notice that due to the properties of tangent lines (radius is perpendicular with tangent line), and the symmetry (the circles are of the same size), the quadrilateral $AXBY$ is actually a square. Therefore, $\overline{AB}$, which is a diagonal, has length $\boxed{4\sqrt{2}}$.

35. The problem statement suggests that the constants can be considered identical. Now we have 4 "e"s and 5 consonants. There are $\binom{9}{4}$ ways to put the "e"s in their places. As the consonants are in either in alphabetical order or reverse we need to multiple by 2 to consider the different cases. This yields a total of $\binom{9}{4}\times 2=\boxed{252}$ rearrangements.

36. To solve for the coordinates of points A and B, we need to solve the equation $x^2+6x+8=-x^2-x+2$. Moving the terms to the left-hand side yields $2x^2+7x+6=0$. The quadratic formula yields roots $x=-2$ and $x=-\frac{3}{2}$. Plugging these x-values back into $y=x^2+6x+8$, we find that the coordinates of A and B are $(-2,0)$ and $\left(-\frac{3}{2},\frac{5}{4}\right)$. Applying the formula for slope gives an answer of $\frac{\frac{5}{4}-0}{-\frac{3}{2}-(-2)}=\boxed{\frac{5}{2}}$.
Note: The order in which you calculate the slope of a line does not affect your answer. You would find the same answer of $\frac{5}{2}$ if you calculated the slope as $\frac{0-\frac{5}{4}}{-2-(-\frac{3}{2})}$, since the minus signs cancel out.

37. We can rewrite all of the roots as pure exponents. The infinite product becomes

$$2^{\frac{1}{1\cdot 2}}\cdot 2^{\frac{1}{2\cdot 3}}\cdot 2^{\frac{1}{3\cdot 4}}\cdots.$$

We can combine all these powers of 2 to find

$$2^{\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots}.$$

We now notice that the exponent telescopes to 1 (as $\frac{1}{1\cdot 2}=\frac{1}{1}-\frac{1}{2}$, $\frac{1}{2\cdot 3}=\frac{1}{2}-\frac{1}{3}$, and so on). Therefore, the infinite product equals $2^1=\boxed{2}$.

38. First, note that by Vieta's Formulas, the product of the roots of a monic, even-degree polynomial is equal to its constant term. In addition, note that the constant term of a polynomial is equal to the value of the polynomial at $x = 0$ (since every term except the constant term becomes 0). The monic polynomial $P(Q(x))$ has degree $3 \cdot 2018 = 6054$, which is even, hence the product of its roots is equal to its constant term, which is $P(Q(0))$. We may evaluate $P(Q(0))$ as $P(Q(0)) = P(2) = 2^{2018} + 2^{2017} + \cdots + 2^2 + 2^1 + 1$. This is a geometric series which simplifies to $2^{2019} - 1$. The smallest power of 2 greater than $2^{2019} - 1$ is 2^{2019}, so the answer is $y = \boxed{2019}$.

39. First, consider a single side of the square. The shape which contains all possible points P which satisfy the initial condition just for that side is an equilateral triangle whose base is that side and whose third vertex lies in the interior of the square. Notice that if we choose P to be outside this region, the equilateral triangle must lie outside the square. The area in question is the intersection of all 4 of these equilateral triangles, which is a convex octagon centered at the center of the square. This octagon is composed of 8 identical 45-60-75 triangles. Using similar triangles and the fact that the side between the 45 and 60 degree angles of one of these triangles is parallel and equidistant from 2 opposite sides of the square, we find that the length of this side is $\frac{3-\sqrt{3}}{6}$. By dropping the altitude from the 75 degree angle, we find that the area of the triangle is $\frac{9-5\sqrt{3}}{24}$, so the area of all 8 triangles is $8 \cdot \frac{9-5\sqrt{3}}{24} = \boxed{\frac{9-5\sqrt{3}}{3}}$.

40. Let Andrew's and Simon's initial starting positions be A and S, respectively. Notice that the described circle is also the locus of all points P such that $\frac{AP}{SP}$ is some constant k, which is also the ratio of Andrew's to Simon's speeds. Consider the points on the locus which are also line $\overleftrightarrow{AS}$. There are two such points, one on segment $\overline{AS}$ and one not on ray $\overrightarrow{SA}$. Subsequently, the segment connecting these two points must be the diameter of the circle (if the segment was a chord, then $\overleftrightarrow{AS}$ wouldn't be a line of symmetry, which doesn't make intuitive sense. We can compute that given a ratio k, the radius of the circle is $r = \frac{24}{k+1} + \frac{24}{k-1}$ ($\frac{48}{k+1}$ is the distance from S to the point on segment $\overline{AS}$ and $\frac{48}{k-1}$ is the distance from S to the point not on ray $\overrightarrow{SA}$). When Andrew runs at 20 miles per hour and Simon runs at 12, $k = \frac{20}{12} = \frac{5}{3} \longrightarrow r = 90$. Setting $r = 45 - 13 = 32$ in the original equation, we get that $k = 2$ (an extraneous negative solution for k is also given). Thus, $\frac{20}{x} = k = 2 \longrightarrow x = \boxed{10}$ miles per hour.