Joe Holbrook Memorial Math Competition

6th Grade

October 14, 2018

General Rules

- You will have **75 minutes** to solve **40 questions**. Your score is the number of correct answers.
- Only answers recorded on the answer sheet will be graded.
- This is an individual test. Anyone caught communicating with another student will be removed from the exam.
- Scores will be posted on the website. Please do not forget your ID number, as that will be the sole means of identification for the scores.
- You may not use the following aids:
  - Calculator or other computing device
  - Compass
  - Protractor
  - Ruler or straightedge

  In addition, you must use the scrap paper supplied by the proctors.

Other Notes

- Write legibly. If the graders cannot read your answer, you will be given no credit for that question.
- Fractions should be written in lowest terms. Please convert all mixed numbers into improper fractions.
- For constants such as e or π, do not approximate your answer: for example, if the answer to a question is 7π, then you should not write 22 or 21.99.
- You do not need to write units in your answers.
- Rationalize all denominators. In addition, numbers within a square root must be squarefree, e.g. $\sqrt{63}$ should be written as $3\sqrt{7}$.
- Ties will be broken by the number of correct responses to questions 31 through 40. Further ties will be broken by the number of correct responses in the last five questions.

1. Members of the BCA Math Team are writing problems. If they write 5 problems every minute, how long, in minutes, will it take for them to write 80 problems?

2. What is the value of $2 \div (0+1) \cdot 8$?

3. After Shalin got his driver's license, his friends gave him a chocolate cake. First, Shalin ate $\frac{1}{3}$ of it. Then, Jenn ate $\frac{2}{5}$ of what remained. What fraction of the cake was left?

4. A quarter of all students taking the JHMMC registered in August. If 130 students registered in August, how many total students are taking the JHMMC?

5. Lunch at the Bergen County Academies cafeteria costs four dollars. There are ten months in the school year, with eighteen school days each. Joe buys lunch (and nothing else) at the Bergen County Academies cafeteria on every school day. How much money, in dollars, will he spend over one entire school year?

6. Every day, a student solves three problems more than she did the day before. If she solved four problems on the first day, how many problems did she solve on the twelfth day?

7. Allison rolls a fair 6-sided die. What is the probability that she rolls a prime number?

8. Mikako the Monkey has a favorite positive number. She multiplies this number by 4 and then subtracts 11. Finally, she takes the square root of the result to get 3. What is Mikako's favorite positive number?

9. A hydra starts out with 20 heads, and grows 3 new heads every time one of its heads is chopped off. How many heads will the hydra have if 7 heads are chopped off?

10. What is the positive difference between the sum of the factors of 2018 and the number of factors of 2019?

11. In how many ways can 2019 be written as the sum of two prime numbers (where order does not matter)?

12. If $a \star b = 5a + 3b + (b \star (a-1))$ and $1 \star 1 = 10$, then what is $2 \star 2$?

13. A bag is filled with 600 Starbursts, 30% of which are pink. How many pink Starbursts must be removed from the bag for 20% of the remaining Starbursts to be pink?

14. How many digits does the number $4^{1010} \cdot 5^{2018}$ have?

15. At a pet store, there are dogs and birds. Haneul and Autumn walk into the store, and Autumn decides to show off her counting skills. If Autumn counts 76 heads and 232 legs, how many dogs are in the shop?

16. An infinite line of dominoes labeled $1, 2, 3 \ldots$ is placed in a line in that order. The domino labeled 1 is knocked over with probability 1. For $n \geq 2$, if the domino labeled $n-1$ is knocked over, then the domino labeled n gets knocked over with probability $\frac{n}{n+1}$. What is the probability that the domino labeled 2018 will be knocked over?

17. Amy played a total of n games and won half of them. If Amy plays 7 more games and wins all of them, she will have won a total of $\frac{3}{5}$ of all of her games. What is the value of n?

18. Andrew the Sun Bear has taken five 100-point math tests. His average score on these five tests is 94. What minimum score must Andrew get on his sixth math test in order to raise his average to 95?

19. At a dance, there are 120 people. Some are paired up so that there are 48 couples. What is the probability that a randomly selected person is not in a couple?

20. Autumn has a peculiar deck of cards. The deck has one 1, two 2's, three 3's, and so on, up to ten 10's. If she randomly selects a card from the deck, what is the probability that it is a 6?

21. The ratio of the side-lengths of two pentagons is $\frac{3}{5}$. The area of the larger pentagon is 35. What is the area of the smaller pentagon?

22. Lyp is taking free throws. He has a 70 percent chance of making any given free throw. Before each shot, he randomly guesses whether he will make the throw, either yelling "I'll make it" or "I'll miss it". What is the probability that he calls exactly 2 out of his next 3 free throws correctly?

23. Find the number of 6-digit positive integers whose digits increase from left to right. For example, 123456 is one such number, but 112344 and 345264 are not.

24. If exactly $\frac{5}{8}$ of the people in a non-empty room like chocolate ice cream, and if $\frac{3}{5}$ of the people in the room like vanilla ice cream, then what is the smallest possible positive number of people in the room who like both chocolate and vanilla ice cream?

25. Ben, Eddie, and Harahm are building a birdhouse. If Ben and Eddie worked together to build the birdhouse, it would take them 3 days. If Ben and Harahm worked together to build the birdhouse, it would take them 4 days. If Harahm and Eddie worked together to build the birdhouse, it would take them 5 days. If all of them worked together to build the birdhouse, how long, in days, would it take them to complete it?

26. A lamb is tied to a post located at the origin of the coordinate plane by a rope that measures 6 units. The farmer who owns the lamb also keeps two wolves tied up at $(6, 6)$ and $(-6, -6)$ with ropes that measure 6 units. What is the area of the region that the lamb can wander in without being in the range of the wolves? (Note: all motion is restricted to the coordinate plane.)

27. A compulsive liar is a person who lies exactly 50% of the time. If a compulsive liar makes 17 statements, what is the probability that more than half of them are true?

28. Seb has an arithmetic sequence with 3000 terms. The common difference of the sequence is $\left(1 + \frac{\pi}{2}\right)^3$, and the sum of the first and last terms is 50. Find the sum of all 3000 terms.

29. In a tournament of 1024 people, there are 256 groups of 4 people, and each person is in exactly one group. Within each group, every possible pair of people plays exactly once, and a single winner moves on to the next round. This procedure is repeated with the winners of each round until a single player remains. How many games take place in the tournament?

30. Let the roots of the polynomial $x^2 + 5x - 17$ be p and q. What is the value of $\frac{p}{q} + \frac{q}{p}$?

31. Two trains, 360 miles apart, begin moving towards each other at 60 miles per hour. The first train moves to the East and the second moves to the West. A crow, flying at 40 miles per hour, is flying between the trains. It begins midway between the trains and starts flying West. Every time the crow encounters a train, it reverses its direction while maintaining its speed. When the trains meet, what total distance has the crow traveled?

32. Let ϕ denote the number of yards to Alpha Centauri. What is the minimum value of

$$\frac{x^2}{2018} + \frac{\phi^2 + 2017}{2018} + \frac{\phi x}{1009},$$

as x varies over the real numbers?

33. Elaine finds herself at the origin and wants to travel to the point $(1, 5)$. On each turn, she can only go one unit up and one unit left OR one unit up and one unit right. If Elaine has an infinite number of turns, in how many ways can she accomplish her goal?

34. Daniel draws the parabola $y = x^2 + 6x + 8$ on the blackboard. Then, Derek draws the parabola $y = -x^2 - x + 2$ on the blackboard. The two parabolas meet at the distinct points A and B. Find the slope of the line AB.

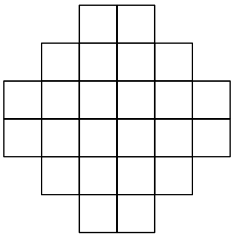
35. Find the number of times the digit 9 appears in following list of numbers:

$$0,\ 1,\ 2,\ 3,\ 4,\ \ldots,\ 9999.$$

36. Evaluate $\sqrt[2!]{2^{0!}}\,\sqrt[3!]{2^{1!}}\,\sqrt[4!]{2^{2!}}\,\sqrt[5!]{2^{3!}}\cdots$.

37. In trapezoid $ABCD$, $AB \parallel CD$ and $AC \perp BD$. Given that $AB = 63$, $CD = 149$, and $\angle DBA = 37.5^\circ$, find the distance between the midpoints of $\overline{AB}$ and $\overline{CD}$.

38. The Aztec Diamond of order n is defined as the lattice of squares whose centers (x, y) satisfy $|x| + |y| \leq n$.



Starting from the leftmost square on the top row of an Aztec Diamond of order 3, find the number of paths that travel along a contiguous path to the rightmost square on the bottom row such that:

- No square is visited twice, and
- moves can only be to the left, right, or down.

39. Call a polynomial *irreducible* if it cannot be factored into two nonconstant polynomials with integer coefficients. The polynomial $x^9 + 243x^3 + 729$ can be factored into two nonconstant irreducible polynomials with positive integer coefficients. Find the factor with the lower degree.

40. Given a polynomial $P(x)$, we define the *period* of a complex number x_0 with respect to $P(x)$ to be the least number k (if it exists) such that

$$\underbrace{P(P(\cdots P}_{k}(x_0)\cdots)) = x_0.$$

Compute the sum of the numbers that have period 2 with respect to the polynomial $P(x) = x^2 + 2x$.