Joe Holbrook Memorial Math Competition

7th Grade

October 14, 2018

General Rules

- You will have **75 minutes** to solve **40 questions**. Your score is the number of correct answers.
- Only answers recorded on the answer sheet will be graded.
- This is an individual test. Anyone caught communicating with another student will be removed from the exam.
- Scores will be posted on the website. Please do not forget your ID number, as that will be the sole means of identification for the scores.
- You may not use the following aids:
  - Calculator or other computing device
  - Compass
  - Protractor
  - Ruler or straightedge

  In addition, you must use the scrap paper supplied by the proctors.

Other Notes

- Write legibly. If the graders cannot read your answer, you will be given no credit for that question.
- Fractions should be written in lowest terms. Please convert all mixed numbers into improper fractions.
- For constants such as e or π, do not approximate your answer: for example, if the answer to a question is 7π, then you should not write 22 or 21.99.
- You do not need to write units in your answers.
- Rationalize all denominators. In addition, numbers within a square root must be squarefree, e.g. $\sqrt{63}$ should be written as $3\sqrt{7}$.
- Ties will be broken by the number of correct responses to questions 31 through 40. Further ties will be broken by the number of correct responses in the last five questions.

1. The BCA Math Team is writing problems. They write 5 problems every minute. How many minutes will it take for them to write 80 problems?

2. What is the value of $2 \div (0 + 1) \cdot 8$?

3. Lunch at the Bergen County Academies cafeteria costs four dollars. There are ten months in the school year, with eighteen school days each. Joe buys lunch (and nothing else) at the Bergen County Academies cafeteria on every school day. How much money, in dollars, will he spend over one entire school year?

4. Tickets to a play cost \$6 dollars each at full-price. A class buys 20 full-price tickets and 3 half-price tickets. How much did the class pay?

5. Mikako the Monkey has a favorite positive number. She multiplies this number by 4 and then subtracts 11. Finally, she takes the square root of the result to get 3. What is Mikako's favorite positive number?

6. Matt loves sushi and spends \$57 for three tuna rolls and one California roll. Matt can also buy two tuna rolls and four California rolls for \$58. How much, in dollars, would Matt have to pay for one tuna roll and one California roll?

7. In how many ways can 2019 be written as the sum of two prime numbers (where order does not matter)?

8. Jake is given a number. He is told to add 7 to it and multiply the result by 4. Instead, he adds 7 multiplied by 4 to the number. He gets 41. What answer should he have gotten?

9. Emi needs to charge her Robotic Wolf. The Wolf starts off with 40 percent battery. The Wolf charges at a rate of 1 percent per minute, and if it is awake, drains battery at the rate of 0.5 percent per minute. As Emi charged the Wolf, it was awake for half the time it was charging. How long does it take for the Wolf to charge to full battery?

10. A bag is filled with 600 Starbursts, of which 30% are pink. How many pink Starbursts must be removed from the bag for 20% of the remaining Starbursts to be pink?

11. Mrs. Icon tells Reina and Erika that models must "strut, strut, strut, pose", but neither of them can get it right. Reina goes "strut, strut, strut, strut, pose" while Erika goes "strut, strut, strut, strut, strut, pose". In a rehearsal, Mrs. Icon, Reina, and Erika all start their patterns at the same time. If each action takes one beat and the pattern starts again after posing, how many beats will it take for all three of them to pose at the same time?

12. An infinite line of dominoes labeled $1, 2, 3 \ldots$ is placed in a line in that order. The domino labeled 1 is knocked over with probability 1. For $n \geq 2$, if the domino labeled $n-1$ is knocked over, then the domino labeled n gets knocked over with probability $\frac{n}{n+1}$. What is the probability that the domino labeled 2018 will be knocked over?

13. x is a positive integer such that $\text{lcm}(4, x) = 100$. What is the sum of all possible values of x?

14. At a dance, there are 120 people. Some are paired up so that there are 48 couples. What is the probability that a randomly selected person is not in a couple?

15. At 12:00 PM, Autumn leaves her home, running at 5 miles per hour. Twelve minutes later, Haneul leaves Autumn's home and starts biking towards her (in the same direction) at 8 miles per hour. How many minutes will it take her to catch up to Autumn?

16. Haneul the Beaver can cut down a tree in 10 minutes. Working with her friend Handel, they can cut down a tree in 6 minutes. How many hours will it take Handel to cut down 24 trees by herself?

17. A triangle has side lengths 5, 12, and 13. What is the length of the altitude to the side of length 13?

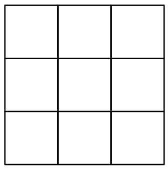
18. The ratio of the side-lengths of two pentagons is $\frac{3}{5}$. The area of the larger pentagon is 35. What is the area of the smaller pentagon?

19. If exactly $\frac{5}{8}$ of the people in a non-empty room like chocolate ice cream, and if $\frac{3}{5}$ of the people in the room like vanilla ice cream, then what is the smallest possible positive number of people in the room who like both chocolate and vanilla ice cream?

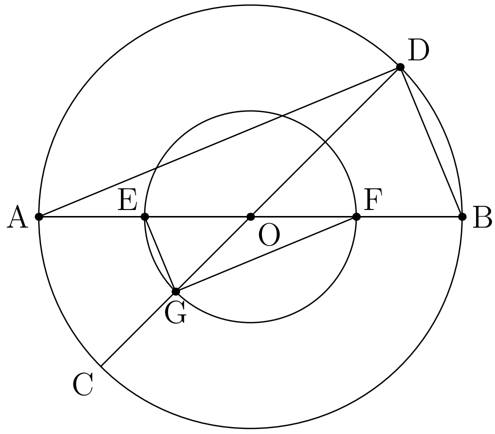
20. At the grocery store, Sebastian only bought apples, whose price is some whole number of cents. He bought at least two apples but strictly less than 30 apples. If Sebastian paid the cashier \$10 and received \$4.61 in change, what is the sum of all possible values for the number of apples Sebastian bought?

21. Seb has an arithmetic sequence with 3000 terms. The common difference of the sequence is $(1 + \frac{\pi}{2})^3$ and the sum of the first and last terms is 50. Find the sum of all 3000 terms.

22. Kelvin the Frog wants to color each of the nine squares of a 3×3 grid one of red, green, or blue so that no row or column has more than one square of a given color. In how many different ways can Kelvin color the grid?



23. Two circles are centered at point O, one with radius 3 and one with radius 5. AB and CD are both diameters of the larger circle. Let points E and F be the intersection points of AB with the smaller circle, and G be an intersection point of CD with the smaller circle (see figure). Find the ratio of the areas of triangle ABD and triangle EFG.



24. How many digits does the number $4^{1010} \cdot 5^{2018}$ have?

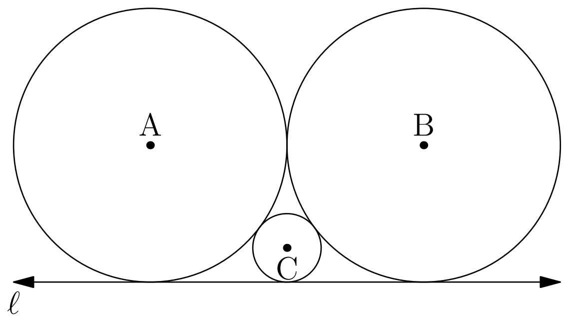
25. Let the roots of the polynomial $x^2 + 5x - 17$ be p and q. What is the value of $\frac{p}{q} + \frac{q}{p}$?

26. Julia, Sonya, Rina, Simran, and Kalina are having a sleepover and want to arrange the mattresses in a way that makes them all happy. Julia and Simran want to sleep next to each other, but Julia and Sonya refuse to sleep next to each other. Further, Rina and Kalina want to sleep next to each other. In how many ways can they arrange the mattresses?

27. Shrek has a magic knife that can cut a slice of cheese in the ratio $1 : x$ for some fixed real $0 < x < 1$. Shrek uses the knife to cut a slice of cheese into two slices. Shrek uses the knife a second time to cut the bigger slice into two more slices. This procedure yields three pieces of cheese. It turns out that when the two smallest pieces combined is identical to the largest piece. Find x.

28. Consider the sequence $a_1, a_2, a_3, \ldots, a_{2018}$, where $a_n = 2n$ for all n. Find the number of pairs of positive integers (l, r) such that $l < r$ and the set $\{a_l, a_{l+1}, \ldots, a_{r-1}, a_r\}$ has median 2019.

29. Circles A and B have radius $\frac{1}{2}$. Circle A is externally tangent to circle B, and both circle A and circle B lie tangent to the same side of the horizontal line ℓ. Let circle C be externally tangent to circle A, circle B, and line ℓ. Find the area of circle C.



30. At camp, Bryan learns a new game called “AiFam” where each player is assigned a character of either good or evil. There is a $\frac{2}{3}$ chance of getting a good character. However, Bryan hates it when he gets a good character, so $\frac{1}{2}$ of the time he's assigned a good character, he'll lie and say he got an evil character. On the other hand, when he's assigned an evil character, $\frac{1}{4}$ of the time he'll lie and say he got a good character. If Bryan starts off a game claiming he got an evil character, what is the probability that he was actually assigned an evil character?

31. Daniel draws the parabola $y = x^2 + 6x + 8$ on the blackboard. Then, Derek draws the parabola $y = -x^2 - x + 2$ on the blackboard. The two parabolas meet at points A and B. Find the slope of the line AB.

32. Adithya is walking on the coordinate plane. At each step, Adithya can only move one unit to the right or one unit up. Given that Adithya starts at the origin, how many ways are there for him to get to $(4, 5)$ without passing through $(2, 2)$?

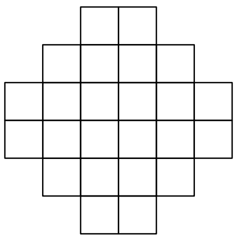
33. Compute the sum
$$\tfrac{1}{1\cdot3} + \tfrac{1}{3\cdot5} + \tfrac{1}{5\cdot7} + \cdots + \tfrac{1}{99\cdot101}.$$

34. Find the number of times the digit 9 appears in the following list of numbers:
$$0,\ 1,\ 2,\ 3,\ 4,\ \ldots,\ 9999$$

35. Let $P(x) = x^{2018} + x^{2017} + \cdots + x^2 + x + 1$ and $Q(x) = x^3 + 2x^2 + 3x + 2$. Let M be the product of the roots of $P(Q(x))$ (with multiplicity). What is the least positive integer y such that $2^y > M$?

36. Simon writes the numbers $1, 2, 3, \ldots, 10$ on a board. At each step, he randomly chooses two numbers a and b written on the board and replaces both of them with the number $ab + 2a + 2b + 2$. Let E be the expected value of the last number written on the board. What is the integer closest to $\frac{13!}{E}$?

37. The Aztec Diamond of order n is defined as the lattice of squares whose centers (x, y) satisfy $|x| + |y| \le n$.



Starting from the leftmost square on the top row of an Aztec Diamond of order 3, find the number of paths that travel along a contiguous path to the rightmost square on the bottom row such that:

- No square is visited twice, and
- moves can only be to the left, right, or down.

38. Call a polynomial *irreducible* if it cannot be factored into two nonconstant polynomials with integer coefficients. The polynomial $x^9 + 243x^3 + 729$ can be factored into two nonconstant irreducible polynomials with positive integer coefficients. Find the factor with the lower degree.

39. Shrek and $n - 1$ $(n \ge 1)$ other ogres take turns playing a game on a 5×10 grid. On each ogre's turn, the ogre must select a piece of the grid with area ≥ 2 (possibly the entire grid) and cut it along one gridline to form two smaller pieces. The last ogre to be able to make a valid move wins. Given that Shrek goes first, for how many values of n does Shrek have a winning strategy?

40. The probability that the triangle constructed from 3 randomly chosen points on a circle contains at least one angle less than θ, where $\theta < 45^\circ$, is $\frac{5}{9}$. What is the value of θ, in degrees?