Joe Holbrook Memorial Math Competition

7th Grade Solutions

October 14, 2018

1. 80 problems $\cdot$ 1 minute/5 problems $= \boxed{16}$ minutes.

2. We use PEMDAS: $2 \div (0+1) \cdot 8 = 2 \div 1 \cdot 8 = 2 \cdot 8 = \boxed{16}$.

3. The answer can be found through multiplication: 10 months $\cdot \frac{18 \text{ days}}{\text{month}} \cdot \frac{4 \text{ dollars}}{\text{day}} = \boxed{720}$ dollars.

4. Half-price tickets cost $\frac{1}{2} \cdot 6 = 3$ dollars each. Therefore, the total cost of the tickets is $6 \cdot 20 + 3 \cdot 3 = \boxed{129}$ dollars.

5. Let x be Mikako's favorite positive number. Then, $\sqrt{4x-11} = 3$. $(\sqrt{4x-11})^2 = 3^2$, so $4x - 11 = 9$. Add 11 on both sides to get $4x = 20$. Then, divide 4 on both sides to get $x = 5$. Thus, Mikako's favorite positive number is $\boxed{5}$.

6. Let t be the price of a tuna roll and c be the price of a California roll. Then

$$\begin{cases} 3t + c = 57 \\ 2t + 4c = 58. \end{cases}$$

   Multiplying the first equation by 4, subtracting the second equation from that, and dividing by 10 gives $t = 17$. Using the first equation, $c = 6$. Thus, it costs $t + c = 17 + 6 = \boxed{23}$ dollars for Matt to buy one tuna roll and one California roll.

7. Since 2019 is odd, it cannot be the sum of two odd prime numbers, so whenever it is the sum of two primes one of these must be 2. In this case $2019 = 2 + 2017$ is the only possibility, so there is $\boxed{1}$ way.

8. We know the $7 \times 4 = 28$, so Jake added 28 to his number to get 41. His originally number must have been 13. Adding 7 to 13 gives 20, and multiplying 20 by 4 gives $\boxed{80}$.

9. Let T be the amount of time the Wolf is awake during charging. Equating percentages, we get $1 \cdot (2T) - 0.5(T) = 100 - 40$, which simplifies to $T = 40$. Thus, the Wolf spent $2T = \boxed{80}$ minutes charging.

10. To begin with, 180 Starbursts are pink and 420 are not pink. By removing pink Starbursts only, there remain exactly 420 Starbursts which are not pink. If this must comprise 80% of the bag, then there must be $\frac{420}{0.8} = 525$ Starbursts at the end, indicating that $\boxed{75}$ were removed.
    Alternatively, one may also use the equation $\frac{180-x}{600-x} = 0.2 \implies x = \boxed{75}$ to represent that after x pink Starbursts have been removed there must be 20% pink Starbursts remaining.

11. The periods of each person's motion are $4, 5, 6$, so if they all pose at the correct time at time t, we have $4, 5, 6 \mid t$. Since t is minimal, the answer is $\operatorname{lcm}(4, 5, 6) = 2 \cdot 2 \cdot 3 \cdot 5 = \boxed{60}$.

12. The probability that a domino (except for the first one) is knocked over is $\frac{n}{n+1}$, and a domino won't be knocked over unless the previous one is knocked over. Therefore, each domino with label ≤ 2018 must be knocked over, so the answer is $\frac{2}{3} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{2018}{2019} = \boxed{\frac{2}{2019}}$.

13. The least common multiple is $100 = 2^2 \cdot 5^2$. Since 4 has no factors of 5, x must be a multiple of $5^2 = 25$. Moreover, x must be a factor of 100. We see that the only multiples of 25 that are factors of 100 are $25, 50, 100$. All of these work, and their sum is $\boxed{175}$.

14. The 48 couples contain $48 \cdot 2 = 96$ people. The probability a randomly selected person is not in a couple is $\frac{120-96}{120} = \frac{24}{120} = \boxed{\frac{1}{5}}$.

15. Haneul bikes 3 more miles than Autumn walks in one hour. Autumn ran one mile in 12 minutes, so it will take Haneul $\frac{1}{3}$ of of an hour, or $\boxed{20}$ minutes to catch up to Autumn.

16. Haneul can cut 6 trees down in one hour. Haneul and Handel together can cut 10 trees down in one hour, so Handel can cut 4 trees down in one hour. Hence, it will take Handel $\frac{24}{4} = \boxed{6}$ hours to cut down 24 trees.

17. Let h be the length of the altitude to the side with length 13. Since this is a 5-12-13 right triangle, its area is $\frac{5\cdot 12}{2} = 30$. However, the area may also be computed as $\frac{13\cdot h}{2}$. Solving the equation $\frac{13\cdot h}{2} = 30$ yields the answer of $h = \boxed{\frac{60}{13}}$.

18. Since the ratio of side-lengths between two pentagons is $\frac{3}{5}$, the ratio of the areas of the two pentagons is $\left(\frac{3}{5}\right)^2 = \frac{9}{25}$. Let the area of the larger pentagon be x. Then

$$\frac{9}{25} = \frac{x}{35} \implies x = 35 \cdot \frac{9}{25} = \boxed{\frac{63}{5}}.$$

19. The number of people in the room must be divisible by 40. If it is $40k$, then $25k$ people like chocolate and $24k$ like vanilla. This means at least $24k + 25k - 40k = 9k$ people must like both; if any fewer like both then the total exceeds $40k$. Thus the smallest number of people is $\boxed{9}$ when $k = 1$. This works if 9 like both, 16 like only chocolate, and 15 like only vanilla.

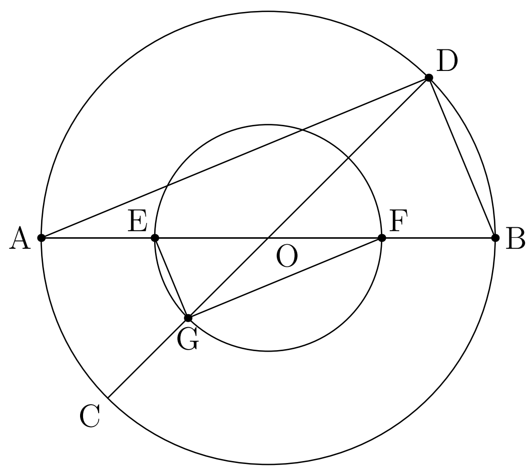
20. Sebastian's total purchase cost $10 - 4.61 = \$5.39$. Since each apple costs a whole number of cents, the number of apples he bought must be a factor of 539 which lies between 1 and 30. Since $539 = 7^2 \cdot 11$, the only such factors are 7 and 11. Therefore, the answer is $7 + 11 = \boxed{18}$.

21. The common difference is irrelevant here. It is well known that the sum of the terms of an arithmetic sequence is the average of the first and last terms multiplied by the number of terms. The average of the first and last terms is their sum divided by 2, which is 25. Therefore, the answer is $25 \cdot 3000 = \boxed{75000}$.

22. Since there are only 3 colors, each row and column must contain one square of each color. There are 3! ways to choose the locations of the red squares such that there is one in every row and column. Next, we see that regardless of the choice of the location of the green square in the first row (among 2 choices), there is a unique way of filling the board. Thus, there are $6 \times 2 \times 1 = \boxed{12}$ ways to color the grid.

23. Let $[P]$ denote the area of figure P. First, note that $\angle EOG = \angle DOB$. Also, $OE = OG = 3$ and $OD = OB = 5$. By SAS similarity, $\triangle EOG \sim \triangle DOB$ with ratio $\frac{3}{5}$. Since the ratio of the areas of similar figures is equal to the square of the ratio of the sides, $[EOG] = \frac{9}{25}[DOB]$. Similarly, $[GOF] = \frac{9}{25}[DOA]$. Then

$$\frac{[ABD]}{[EFG]} = \frac{[DOB] + [DOA]}{[EOG] + [GOF]} = \frac{[DOB] + [DOA]}{\frac{9}{25}[DOB] + \frac{9}{25}[DOA]} = \boxed{\frac{25}{9}}.$$



24. $4^{1010} \cdot 5^{2018} = 2^{2020} \cdot 5^{2018} = 4 \cdot 10^{2018}$ has $\boxed{2019}$ digits.

25. The sum $\frac{p}{q} + \frac{q}{p}$ can be rewritten as $\frac{p^2+q^2}{pq} = \frac{(p+q)^2-2pq}{pq}$. Using Vieta's Formulas, $p+q = -5$ and $pq = -17$. Plugging these values into the above fraction yields the answer of $\frac{(p+q)^2-2pq}{pq} = \frac{25+34}{-17} = \boxed{-\frac{59}{17}}$.

26. Since Julia's mattress has to be next to Simran's, we can consider the two mattresses as one block. Within that block, there are 2 ways to arrange the mattresses. The same reasoning applies to Rina's and Kalina's mattresses. Thus, ignoring the fact that Julia refuses to sleep next to Sonya, there are $2 \cdot 2 \cdot 3! = 24$ ways to order the five people. Now, we must subtract the arrangements in which Sonya is next to Julia. Here, we must modify our two blocks: there must be one with Julia, Simran, and Sonya, and there must be another with Rina and Kalina. Within the 3-mattress block, there are 2 posible arrangements, since Julia has to be between Simran and Sonya. In the 2-mattress block, there are also 2 possible arrangements. Therefore, there are $2 \cdot 2 \cdot 2! = 8$ arrangements in which Julia is next to Sonya. Thus, the final answer is $24 - 8 = \boxed{16}$.

27. Let the original slice of cheese have size 1. Then the sizes of the three cut-up pieces, in decreasing order of size, are $\frac{1}{(x+1)^2}$, $\frac{x}{(x+1)^2}$, and $\frac{x}{x+1}$. Setting the former equal to the sum of the latter two yields the equation

$$\frac{1}{(x+1)^2} = \frac{x}{(x+1)^2} + \frac{x}{x+1} \implies x^2 + 2x - 1 = 0 \implies x = \boxed{\sqrt{2} - 1},$$

as $x > 0$.

Alternative Solution: Since the sum of the sizes of the two smaller pieces is equal to the size of the largest piece, it follows that the largest piece has size $\frac{1}{2}$. Therefore, $\frac{1}{(x+1)^2} = \frac{1}{2} \implies x = \sqrt{2} - 1$, as above.

28. The set $\{a_l, a_{l+1}, \ldots, a_{r-1}, a_r\}$ is an arithmetic sequence, meaning that its median is the average of the first and last elements. This is $\frac{a_l + a_r}{2} = \frac{2l + 2r}{2} = l + r = 2019$. l must be at least 1 and at most 1009 (since $l < r$), so the number of possible pairs is $\boxed{1009}$.

Alternatively, note that $l \leq 1009$ and by symmetry $r = 2019 - l$ is fixed, so each of the 1009 choices of l yields exactly one pair, as before.

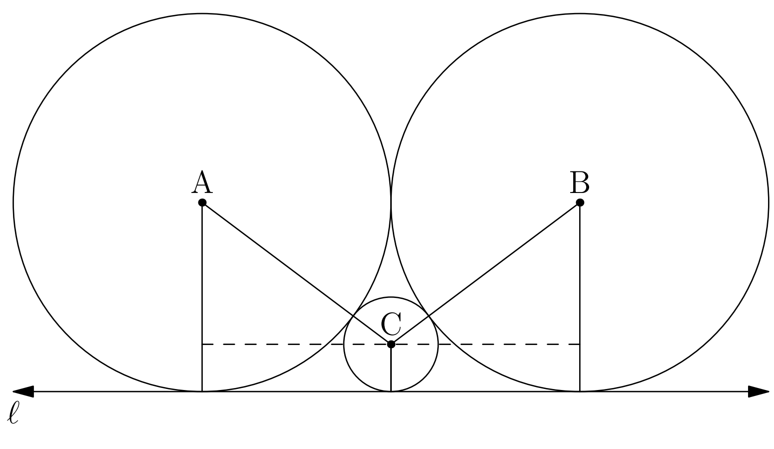
29. Let the radius of circle C be denoted by r. Let circles A, B, and C have centers O_A, O_B, and O_C, respectively. Let the tangency point of circle B with line ℓ be Y, and let the tangency point of circle C with line ℓ be Z. Furthermore, let P be the foot of the perpendicular from O_C to segment O_BY. The length of segment O_BP equals $O_BY - PY = O_BY - O_CZ = \frac{1}{2} - r$. In addition, the length of segment O_CP equals the length of ZY, which equals $\frac{1}{2} \cdot O_AO_B = \frac{1}{2}$. Finally, the length of segment O_BO_C equals the sum of the radii of circles B and C, which is $r + \frac{1}{2}$.
Applying the Pythagorean Theorem to right triangle O_CPO_B yields

$$\left(\frac{1}{2} + r\right)^2 = \left(\frac{1}{2} - r\right)^2 + \left(\frac{1}{2}\right)^2.$$

Therefore, by the Difference of Squares Identity,

$$\left(\frac{1}{2} + r\right)^2 - \left(\frac{1}{2} - r\right)^2 = (2r) \cdot (1) = \frac{1}{4} \implies r = \frac{1}{8}.$$

Hence, the area of circle C is $\pi r^2 = \pi \left(\frac{1}{8}\right)^2 = \boxed{\frac{\pi}{64}}$.
Note: Imagine that you continued this process by constructing a circle D be externally tangent to circle C, circle B and line ℓ, then a circle E externally tangent to circle D, circle B, and line ℓ, and so on. This process generates some of the **Ford Circles.** The Ford Circles are intimately tied to Farey Sequences, which are of great importance in number theory.



30. This is a problem involving conditional probability. Let A be the event that Bryan gets an evil character, and let B be the event that Bryan says he gets an evil character. We wish to compute $P(A|B)$. By the formula, $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$. We compute $P(A \cap B) = \frac{1}{3}(1 - \frac{1}{4}) = \frac{1}{3}\frac{3}{4} = \frac{1}{4}$. $P(B)$ is the sum of this and the probability of Bryan getting a good character but saying he has an evil character ($P(\overline{A} \cap B)$), which is $\frac{2}{3}\frac{1}{2} = \frac{1}{3}$. Thus, $P(B) = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$ and $P(A|B) = \frac{1}{4} \div \frac{7}{12} = \boxed{\frac{3}{7}}$.

31. To solve for the coordinates of points A and B, we need to solve the equation $x^2 + 6x + 8 = -x^2 - x + 2$. Moving the terms to the left-hand side yields $2x^2 + 7x + 6 = 0$. The quadratic formula yields roots $x = -2$ and $x = -\frac{3}{2}$. Plugging these x-values back into $y = x^2 + 6x + 8$, we find that the coordinates of A and B are $(-2, 0)$ and $\left(-\frac{3}{2}, \frac{5}{4}\right)$. Applying the formula for slope gives an answer of $\frac{1.25-0}{-1.5-(-2)} = \boxed{\frac{5}{2}}$.

32. Denote each move up by U and each move to the right by R. Then, the path can be written as a sequence of Us and Rs. In order to find the number of paths, we can just find the number of sequences, which can be done by choosing where the Us are placed. The total number of ways to go from $(0, 0)$ to $(4, 5)$ is then $\binom{9}{4} = 126$ since there are 9 total steps and 4 moves up. The number of ways through $(2, 2)$ is the number of ways from $(0, 0)$ to $(2, 2)$ multiplied by the number of ways from $(2, 2)$ to $(4, 5)$. This is $\binom{4}{2}\binom{5}{3} = 60$. The number of ways avoiding $(2, 2)$ is thus $126 - 60 = \boxed{66}$.

33. All of the terms are of the form $\frac{1}{n(n+2)}$. We see that

$$\frac{1}{n(n+2)} = \frac{1}{2} \cdot \frac{(n+2) - n}{n(n+2)} = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right).$$

Rewriting each of the terms in this way, the sum becomes

$$\frac{1}{2}\left[\left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{3} + \frac{1}{5}\right) + \cdots + \left(\frac{1}{99} - \frac{1}{101}\right)\right]$$

. This sum telescopes to $\frac{1}{2}\left(1 - \frac{1}{101}\right) = \boxed{\frac{50}{101}}$.

34. We can treat each number as a string of digits, filling in leading zeroes so that all the numbers have 4 digits (since we only care about the digit 9). Then, we want to count the number of 9's in the list 0000, 0001, 0002, ... , 9999. Now, note that all the strings of the form $\overline{abcd}$ where a, b, c, d are base 10 digits are present in the list. Also, the number of times each digit appear in the list must be equal. Since there are 10000 strings of 4 digits each, there are 40000 total digits and the digit 9 must appear $\boxed{4000}$ times.

35. First, note that by Vieta's Formulas, the product of the roots of a monic, even-degree polynomial is equal to its constant term. In addition, note that the constant term of a polynomial is equal to the value of the polynomial at $x = 0$ (since every term except the constant term becomes 0). Since $P(Q(x))$ is monic and has degree $3 \cdot 2018$, the product of its roots is equal to its constant term, which is $P(Q(0))$. We may evaluate $P(Q(0))$ as $P(Q(0)) = P(2) = 2^{2018} + 2^{2017} + \cdots + 2^2 + 2^1 + 1$. This is a geometric series which simplifies to $2^{2019} - 1$. The smallest power of 2 greater than $2^{2019} - 1$ is 2^{2019}, so the answer is $y = \boxed{2019}$.

36. We see $ab + 2a + 2b$ and think of completing the rectangle (a.k.a Simon's Favorite Factoring Trick): $ab + 2a + 2b + 2 = ab + 2a + 2b + 4 - 2 = (a + 2)(b + 2) - 2$. If the first two numbers that Simon chooses are a and b, then the first number he writes down is $(a+2)(b+2)-2$. If he chooses that number and some other number c, then he erases those two and writes down $((a+2)(b+2)-2+2)(c+2)-2 = (a+2)(b+2)(c+2)-2$. This pattern continues for any number of operations. When only one number is written on the board, that number will therefore be $E = (1 + 2)(2 + 2) \ldots (10 + 2) - 2 = \frac{12!}{2} - 2$ regardless of the order that Simon chooses. Then $\frac{13!}{E} = \frac{13!}{\frac{12!}{2} - 2} = \frac{26}{1 - \frac{4}{12!}}$, and the integer closest to this fraction is clearly $\boxed{26}$.

37. For each square, write the number of ways for which it is possible to reach that square. The top row will be 1 1, and for each subsequent row until the middle, since we can choose to come down on each cell, the numbers along the row will be equal to the product of the number in the previous row and the number of squares in that row. A similar statement holds after the middle row, except we can't come down from the edge squares. A quick computation now yields $1 \times 2 \times 4 \times 6 \times 4 \times 2 = \boxed{384}$ ways to get to the bottom right square.

		1	1		
	2	2	2	2	
8	8	8	8	8	8
48	48	48	48	48	48
	192	192	192	192	
		384	384		

38. We recognize the powers of three in the problem, and rewrite the given polynomial as $x^9 + 3^5x^3 + 3^6$. As the first and last terms are both cubes, we try to "complete the cube". Lo and behold, $(x^3 + 9)^3 = x^9 + 27x^6 + 243x^3 + 729$. Therefore, our polynomial is simply $(x^3 + 9)^3 - 27x^6$. This expression is a difference of cubes, which factors as $(x^3 - 3x^2 + 9)q(x)$, for some polynomial $q(x)$ of degree 6. Hence, the answer is $\boxed{x^3 - 3x^2 + 9}$.

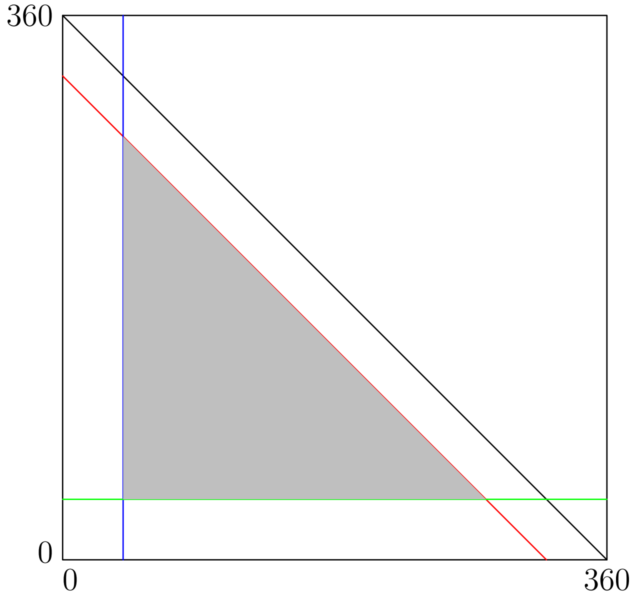
39. We make two observations:

    - After each ogre makes a move, the number of pieces on the board increases by 1.
    - There is initially 1 piece, and the game terminates when there are 50 pieces on the board.

    Thus 49 cuts must be made regardless of strategy, and since Shrek goes first, we need $49 \equiv 1 \mod n$. Thus $n \mid 48$ and there are $\tau(48) = \tau(2^4 3^1) = \boxed{10}$ values of n that work.

40. Call the triangle $\triangle ABC$, and WLOG fix point A arbitrarily on the circle. Then let x denote the measure of arc $\widehat{AB}$ and y denote the measure of arc $\widehat{BC}$. We know that arc $\widehat{AC}$ has to be positive, so $x + y < 360$. Moreover, any choice of x and y that satisfies $x + y < 360$ defines a unique triangle (the black line below). Therefore, the total area of valid triangles is the area of a 360 by 360 triangle isosceles right triangle. To find the desired region, we instead calculate the probability that the triangle has all angles greater than θ. In order for angle A to be greater than θ, arc $\widehat{BC}$ has to be greater than 2θ by the Inscribed Angle Theorem. Thus, $y > 2\theta$ (green line below). Similarly, we draw the lines $x > 2\theta$ and $360 - x - y > 2\theta$ (blue and red lines, respectively). The lower right intersection point of the red line and green line is at $(360 - 4\theta, 2\theta)$. Therefore, the triangle defined by these three lines has legs of length $360 - 6\theta$. Remember that this is the complement of what we actually want to count, so its area should be $1 - \frac{5}{9} = \frac{4}{9}$ of the total area. Thus, we want

$$\frac{\frac{(360^\circ - 6\theta)^2}{2}}{\frac{(360^\circ)^2}{2}} = \frac{4}{9} \implies \theta = \boxed{20^\circ}.$$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