Joe Holbrook Memorial Math Competition

8th Grade

October 14, 2018

General Rules

- You will have **75 minutes** to solve **40 questions**. Your score is the number of correct answers.
- Only answers recorded on the answer sheet will be graded.
- This is an individual test. Anyone caught communicating with another student will be removed from the exam.
- Scores will be posted on the website. Please do not forget your ID number, as that will be the sole means of identification for the scores.
- You may not use the following aids:
  - Calculator or other computing device
  - Compass
  - Protractor
  - Ruler or straightedge

  In addition, you must use the scrap paper supplied by the proctors.

Other Notes

- Write legibly. If the graders cannot read your answer, you will be given no credit for that question.
- Fractions should be written in lowest terms. Please convert all mixed numbers into improper fractions.
- For constants such as e or π, do not approximate your answer: for example, if the answer to a question is 7π, then you should not write 22 or 21.99.
- You do not need to write units in your answers.
- Rationalize all denominators. In addition, numbers within a square root must be squarefree, e.g. $\sqrt{63}$ should be written as $3\sqrt{7}$.
- Ties will be broken by the number of correct responses to questions 31 through 40. Further ties will be broken by the number of correct responses in the last five questions.

1. What is the value of $2 \div (0+1) \cdot 8$?

2. After Shalin got his driver's license, his friends gave him a chocolate cake. First, Shalin ate $\frac{1}{3}$ of it. Then, Jenn ate $\frac{2}{5}$ of what remained. What fraction of the cake was left?

3. Every day, a student solves three problems more than she did the previous day. On the first day, she solved four problems. How many problems did she solve on the twelfth day?

4. To get to school from home, Jason walks 8 meters west then 15 meters south. How many fewer meters would Jason walk if he walked in a straight line from home to school?

5. Gregory, a monster, terrorizes children in his free time. Out of the kindness of his heart, however, he refuses to terrorize any children who are two years old or younger. He will terrorize any other child he encounters. One day, he encountered ten children of the following ages: 1, 1, 2, 3, 4, 7, 8, 9, 11, and 12. Determine the median age of the children he terrorizes.

6. Jake is told to add 7 to a given number and multiply the result by 4. Instead, he adds 7 multiplied by 4 to the number and gets 41. What answer should he have gotten?

7. If $a \star b = 5a + 3b + (b \star (a-1))$ and $1 \star 1 = 10$, then what is $2 \star 2$?

8. Mrs. Icon tells Reina and Erika that models must "strut, strut, strut, pose", but neither of them can get it right. Reina goes "strut, strut, strut, strut, pose" while Erika goes "strut, strut, strut, strut, strut, pose". In a rehearsal, Mrs. Icon, Reina, and Erika all start their patterns at the same time. If each action takes one beat and the pattern starts again after all three posing, how many beats will it take for all three of them to pose at the same time again?

9. Autumn is stuck in traffic because there are cows crossing the road! The cows cross the road one at a time and a different cow crosses immediately after the one in front of it. Adult cows take 2 minutes to cross, but baby cows take 5 minutes. If Autumn had to wait exactly 38 minutes for all the cows to cross and she saw exactly 10 cows while she was waiting, how many baby cows did she see (each cow is either an adult or baby)?

10. Alan's calculator only has the operations of multiplying by 2 and adding 1 to the number on the screen. His calculator starts with the number 2. What is the minimum number of operations that Alan has to use to get to the number 35?

11. Haneul the Beaver is running late. She usually leaves her home at 7:35 AM and arrives to work at 7:55 AM. She works at a nearby dam five miles away from her home. However, she woke up late today and left for work at 7:45 AM. She must get to work by 8:00 AM. At least how much faster (in miles per hour) must Haneul travel in order to get to work on time?

12. Find the number of 6-digit positive integers whose digits increase from left to right. For example, 123456 is one such number, but 112344 and 345264 are not.

13. Jessica wants to avoid running into Samotha after school, so she needs to change her route. Before, she would go 30 feet north, 60 feet east, and then another 50 feet north. Now, she has to go 40 feet south, 30 feet west, 120 feet north, and 90 feet east. If Jessica was able to just make the bus walking at 2 feet per second along her previous route, at what speed (in feet per second) must she walk at along her new route in order to just make the bus?

14. A compulsive liar is a person who lies exactly 50% of the time. If a compulsive liar makes 17 statements, what is the probability that more than half of them are true?

15. Triangle ABC has altitude AH (so H lies on BC). Given that $AB = 13$, $AH = 12$, and $BC = 14$, find AC.

16. Seb has an arithmetic sequence with 3000 terms. The common difference of the sequence is $\left(1 + \frac{\pi}{2}\right)^3$, and the sum of the first and last terms is 50. Find the sum of all 3000 terms.

17. Sebastian is pulling letters out of a bag with 1 A, 2 B's, and 3 C's. Once he pulls out a letter he does not replace it. What is the probability that his first three draws are "B," "C," and then "A," in that order?

18. How many digits long is the number $4^{1010} \cdot 5^{2018}$?

19. Eric wants to take a screenshot of how long he's been facetiming his friend, but being the mathematician he is, he refuses to take the picture unless the time is a palindrome and has at least one 5 in it. The app displays the time in the form of H:MM:SS; for example, the time of 3 hours 7 minutes and 21 seconds would be displayed as 3:07:21. Unfortunately, Eric just missed screenshotting at 3:45:43. How many more seconds do the two friends have to talk until Eric can take a screenshot that satisfies his two requirements?

20. Let the roots of the polynomial $x^2 + 5x - 17$ be p and q. What is the value of $\frac{p}{q} + \frac{q}{p}$?

21. Two trains, 360 miles apart, begin moving towards each other at 60 miles per hour. The first train moves to the East and the second moves to the West. A crow, flying at 40 miles per hour, is flying between the trains. It begins midway between the trains and starts flying West. Every time the crow encounters a train, it reverses its direction while maintaining its speed. When the trains meet, what total distance has the crow traveled?

22. Elaine finds herself at the origin and wants to travel to the point $(1, 5)$. On each turn, she can only go one unit up and one unit left OR one unit up and one unit right. If Elaine has an infinite number of turns, in how many ways can she accomplish her goal?

23. Robot Greg has a robotic arm made up of 3 segments: one of length 4, one of length 2, and one of length 1. His robot arm is connected to his pet Lyp. One end of the segment of length 4 is connected to the point $(0, 0, 0)$ and its other end is connected to one end of the segment of length 2. The other end of the segment of length 2 is connected to one end of the segment of length 1, and the other end of the segment of length 1 is connected to his Lyp. At each connection between two segments, there is a freely-rotating joint that can make any bend in 3 dimensions. In addition, there is a freely-rotating joint at $(0, 0, 0)$. What is the volume of the set of all points the Lyp can visit in 3-D space?

24. How many ways are there to rearrange the letters of "seventeen" so that the consonants are either in alphabetical order or in reverse alphabetical order?

25. Daniel draws the parabola $y = x^2 + 6x + 8$ on the blackboard. Then, Derek draws the parabola $y = -x^2 - x + 2$ on the blackboard. The two parabolas meet at points A and B. Find the slope of the line AB.

26. How many trailing zeros are in the base 10 expansion of $\binom{50}{12}$?

27. Edwin is trying to sleep, but his cat, Oreo, is keeping him awake. She meows every 20 minutes, purrs every 30 minutes, and sneezes every 50 minutes (and yes, she can make more than one sound at the same time). Edwin is very sensitive, so he will wake up from any sound. Assuming that she meows, purrs, and sneezes at midnight, and then continues as in the above pattern, between 11:59 pm and 5:01 am, how many times will Edwin wake up?

28. Kelvin the Frog is at the point $(3, 3)$ on a coordinate plane, and is trying to return to BCA, located at the origin. When he is at a point (x, y), he can only hop to the point $(x - 1, y)$ or $(x, y - 1)$. He can also "superhop" once during his journey, where he goes to the point $(x - 2, y + 1)$. How many ways can he reach BCA if he superhops exactly once?

29. Scalene triangle ABC is inscribed in a circle. Segment BC is extended past C to form a ray. This ray and the line tangent to the circle at point A intersect at point D. Given that $\measuredangle ADC = 32°$ and $\measuredangle CAB = 74°$, find the measure of $\angle ABC$.

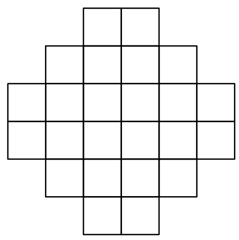
30. Let $P(x) = x^{2018} + x^{2017} + \cdots + x^2 + x + 1$ and $Q(x) = x^3 + 2x^2 + 3x + 2$. Let M be the product of the roots of $P(Q(x))$ (with multiplicity). What is the least positive integer y such that $2^y > M$?

31. Simon is given a 2 by 10 rectangle. He wants to tile every unit square in the rectangle without overlaps by using only 1 by 2 dominoes. How many ways can he tile this rectangle?

32. In trapezoid $ABCD$, $AB \parallel CD$ and $AC \perp BD$. Given that $AB = 63$, $CD = 149$, and $\angle DBA = 37.5°$, find the distance between the midpoints of $\overline{AB}$ and $\overline{CD}$.

33. The faces of a cube are labeled with distinct numbers from 1 to 6. How many ways are there to label the faces such that there is at least one pair of opposite faces whose labels sum to 7? (rotations are considered the same).

34. The Aztec Diamond of order n is defined as the lattice of squares whose centers (x, y) satisfy $|x| + |y| \leq n$.



Starting from the leftmost square on the top row of an Aztec Diamond of order 3, find the number of paths that travel along a contiguous path to the rightmost square on the bottom row such that:

- No square is visited twice, and
- moves can only be to the left, right, or down.

35. Call a polynomial *irreducible* if it cannot be factored into two nonconstant polynomials with integer coefficients. The polynomial $x^9 + 243x^3 + 729$ can be factored into two nonconstant irreducible polynomials with positive integer coefficients. Find the factor with the lower degree.

36. Given a polynomial $P(x)$, we define the *period* of a complex number x_0 with respect to $P(x)$ to be the least number k (if it exists) such that

$$\underbrace{P(P(\cdots P}_{k}(x_0)\cdots)) = x_0.$$

Compute the sum of the complex numbers that have period 2 with respect to the polynomial $P(x) = x^2 + 2x$.

37. Andrew the Sun Bear and Simon are 48 miles apart on an infinite flat plane. At the same time, they start running in a straight line in a random direction, with Andrew running at 20 miles per hour and Simon running at 12 miles per hour. The locus of all possible points where Andrew and Simon could collide is a circle. If Simon had instead run at x miles per hour, then the radius of this circle would be 13 miles smaller. Find the value of x.

38. Shrek and $n - 1$ $(n \geq 1)$ other ogres take turns playing a game on a 5×10 grid. On each ogre's turn, the ogre must select a piece of the grid (possibly the entire grid) and cut it along one gridline to form two smaller pieces. The last ogre to make a valid move wins. Given that Shrek goes first, for how many values of n does Shrek have a winning strategy?

39. The probability that the triangle constructed from 3 randomly chosen points on a circle contains at least one angle less than θ, where $\theta < 45^\circ$, is $\frac{5}{9}$. What is the value of θ, in degrees?

40. Consider the following congruence in $(x_1, \cdots, x_6)$:

$$a_1x_1 + a_2x_2 + \cdots + a_6x_6 \equiv 0 \mod 6,$$

where $a_1, a_2, \cdots, a_6$ are chosen uniformly at random from the set $\{1, 2, 3, 4, 5, 6\}$. Over all choices of $(a_1, \cdots, a_6)$, the minimum number of solutions to the congruence is S. Find the probability that there are S solutions to the congruence.