Joe Holbrook Memorial Math Competition

8th Grade Solutions

October 14, 2018

1. We use PEMDAS: $2 \div (0+1) \cdot 8 = 2 \div 1 \cdot 8 = 2 \cdot 8 = \boxed{16}$.
2. Jenn ate $\frac{2}{3} \cdot \frac{2}{5} = \frac{4}{15}$ of the cake. Therefore, Shalin and Jenn toghether ate $\frac{4}{15} + \frac{1}{3} = \frac{9}{15}$ of the cake. Thus, $1 - \frac{9}{15} = \frac{6}{15} = \boxed{\frac{2}{5}}$ of the cake remained.
3. On day n, the student solves $3n+1$ problems, since on day 1 she solved $3 \times 1 + 1 = 4$ problems and she solves three more problems than she did the day before. On day 12, she solves $3 \times 12 + 1 = \boxed{37}$ problems.
4. The straight path from home to school is the hypotenuse of a right triangle with legs 8 and 15. Using the Pythagorean theorem, the hypotenuse has a length of $\sqrt{8^2 + 15^2} = 17$. If he walked west then south, Jason would walk 23 meters, so Jason would walk $23 - 17 = \boxed{6}$ meters less if he just walked in a straight path from home to school.
5. We should only consider the ages of the children who are three years old or older. These ages are 3, 4, 7, 8, 9, 11, and 12. Through inspection, we conclude that the median of this set is $\boxed{8}$.
6. We know the $7 \times 4 = 28$, so Jake added 28 to his number to get 41. His originally number must have been 13. Adding 7 to 13 gives 20, and multiplying 20 by 4 gives $\boxed{80}$.
7. From the formula, we have
$$2 \star 1 = 5(2) + 3(1) + 1 \star 1 = 23 \implies 2 \star 2 = 5(2) + 3(2) + 2 \star 1 = \boxed{39}.$$
8. This is a problem about LCMs. We're looking for the LCM of 4, 5, and 6, which is $2 \cdot 2 \cdot 3 \cdot 5 = \boxed{60}$.
9. Let a be the number of adult cows and b be the number of babies. Then, we can set up a system of linear equations with the given information. Since Autumn waits for 38 minutes, $2a + 5b = 38$, and since she sees 10 cows, $a + b = 10$. We then use elimination to solve for a and b. We multiply the second equation by 2, which yields $2a + 2b = 20$, and subtract it from the first equation, which yields $3b = 18$, so $b = \boxed{6}$.
10. Initially, it may appear that repeatedly multiplying by 2 at the outset gives the optimal number of operations. Using this strategy, the sequence is 2, 4, 8, 16, 32, 33, 34, 35. However, this strategy requires the addition of three 1s. Instead, we can add 1 before the last multiplication by 2, yielding 2, 4, 8, 16, 17, 34, 35. Thus, a minimum of $\boxed{6}$ operations are needed.
11. It usually takes Haneul 20 minutes to travel five miles, so she travels at a speed of 15 miles per hour. On this particular day, she has at most 15 minutes to travel five miles, so she must travel at a minimum speed of 20 miles per hour, which is $\boxed{5}$ miles per hour faster than usual.
12. There are nine digits to choose from. (There cannot be a zero in the number because, by the condition, zero would have to be the first digit, and a number cannot begin with zero.) For each subset of six digits from 1 through 9, there is exactly one way to arrange them in increasing order. Thus, there are $\binom{9}{6} = \boxed{84}$ 6-digit numbers with their digits in increasing order.
13. The distance Jessica traveled on her old route was $30 + 60 + 50 = 140$ feet, while the distance of Jessica will travel on her new route is $40 + 30 + 120 + 90 = 280$ feet. Therefore, Jessica must double her speed to $\boxed{4}$ feet per second.
14. The probability that the compulsive liar tells more truths than lies is the same as the the probability that he tells more lies than truths. Note that there cannot be an equal number of lies and truths, since there is an odd number of questions in total. Therefore, the desired probability is $\boxed{\frac{1}{2}}$.

15. We know ABH is a right triangle, with hypotenuse 13 and one leg of length 12, so $BH = 5$. Since $BC = 14$, we now know $CH = 9$, and since ACH is also a right triangle, we have $AC = \sqrt{9^2 + 12^2} = \boxed{15}$.

16. The common difference is irrelevant! It's well known that the sum of the terms of an arithmetic sequence is the average of the first and last terms multiplied by the number of terms. In this case, the average of the first and last terms (their sum divided by 2) is 25. Therefore, the answer is $25 \cdot 3000 = \boxed{75000}$.

17. The probability of pulling out a B initially is $\frac{2}{6}$. If he does this successfully, there will be 1 A, 1 B, and 3 C's left in the bag. The probability of pulling out a C after that is $\frac{3}{5}$. If he does *this* successfully, he will have 1 A, 1 B, and 2 C's left, and the probability of pulling out an A is $\frac{1}{4}$. Therefore the combined probability is $\frac{2}{6} \cdot \frac{3}{5} \cdot \frac{1}{4} = \boxed{\frac{1}{20}}$.

18. $4^{1010} \cdot 5^{2018} = 2^{2020} \cdot 5^{2018} = 4 \cdot 10^{2018}$ which has $\boxed{2019}$ digits.

19. The smallest palindrome containing a 5 which is greater than 34543 is 35053. (This can be found by generating palindromes in increasing order: start by incrementing the middle digit, then the tens thousands digits, and so on.) If we write this palindrome in the form displayed by the app, we find that the next time at which Eric can take a screenshot is 3:50:53. This is 5 minutes and 10 seconds after 3:45:43, which yields the answer of $\boxed{310}$ seconds.

20. The sum $\frac{p}{q} + \frac{q}{p}$ can be rewritten as $\frac{p^2+q^2}{pq} = \frac{(p+q)^2-2pq}{pq}$. By Vieta's Formulas, $p + q = -5$ and $pq = -17$. Plugging these values into the above fraction yields the answer of

$$\frac{(p+q)^2-2pq}{pq} = \frac{25+34}{-17} = \boxed{-\frac{59}{17}}.$$

21. Let t be the amount of time it takes (in hours) for the trains to meet. Then $360 = 60t + 60t \implies t = \frac{360 \text{ miles}}{120 \text{ miles/hour}} = 3$ hours. Therefore, the crow flies for 3 hours before the trains meet. Hence, the crow travels a total distance of 40 miles/hour $\times$ 3 hours $= \boxed{120 \text{ miles}}$ before the trains meet.

22. Call moving up one unit and left one unit a *type I move,* and call moving up one unit and right one unit a *type II move.* Then, Elaine must make 2 type I moves and 3 type II moves. She can make these moves in any order. Out of her 5 total moves, 2 of them must be of type I, so there are a total of $\binom{5}{2} = \boxed{10}$ ways she can move to $(1, 5)$.

23. Let x denote the distance between the Lyp and the origin. By the Generalized Triangle Inequality, $1 + 2 + x \geq 4 \implies x \geq 1$. In addition, the Generalized Triangle Inequality impies that $x \leq 1 + 2 + 4 = 7$. It follows that Lyp can reach any point that is between 1 unit and 7 units away from the origin. Therefore, the set of all points he can reach is the interior of a sphere of radius 7 with a sphere of radius 1 removed from its center. Thus, the volume of the set of all points he can reach is

$$\frac{4 \cdot \pi \cdot 7^3}{3} - \frac{4 \cdot \pi \cdot 1^3}{3} = \frac{4 \cdot \pi \cdot (7^3 - 1^3)}{3} = \boxed{456\pi}.$$

24. This means that the constants can be considered identical. Now we have 4 "e"s and 5 consonants. There are $\binom{9}{4}$ ways to put the "e"s in their places. As the consonants are in either in alphabetical order or reverse we need to multiple by 2 to consider the different cases. This yields a total of $\binom{9}{4} \times 2 = \boxed{252}$ rearrangements.

25. To solve for the coordinates of points A and B, we need to solve the equation $x^2 + 6x + 8 = -x^2 - x + 2$. Moving the terms to the left-hand side yields $2x^2 + 7x + 6 = 0$. The quadratic formula yields roots $x = -2$ and $x = -\frac{3}{2}$. Plugging these x-values back into $y = x^2 + 6x + 8$, we find that the coordinates of A and B are $(-2, 0)$ and $\left(-\frac{3}{2}, \frac{5}{4}\right)$. Applying the formula for slope gives an answer of $\frac{5/4-0}{-3/2-(-2)} = \boxed{\frac{5}{2}}$.

26. Finding the number of trailing zeros of a number is equivalent to finding the number of times we can divide 10 into this number. Since $10 = 2 \cdot 5$, we can do this by determining how many factors of 2s and 5s this number has.

    We write $\binom{50}{12}$ as $\frac{50!}{12!38!}$. To find out how many factors of 2 divide this expression, we can find the number of factors of 2 in the denominator, and subtract this from the number of factors of 2 in the numerator. 50! is divisible by $\lfloor 50/2 \rfloor$ factors of 2, $\lfloor 50/4 \rfloor$ factors of 4, which has another factor of 2 in addition to the first, and so on. Thus, 50! has $\lfloor 50/2 \rfloor + \lfloor 50/4 \rfloor + \lfloor 50/8 \rfloor + \lfloor 50/16 \rfloor + \lfloor 50/32 \rfloor = 25 + 12 + 6 + 3 + 1 = 47$ factors of 2. Similarly, 12! has $\lfloor 12/2 \rfloor + \lfloor 12/4 \rfloor + \lfloor 12/8 \rfloor = 6 + 3 + 1 = 10$ factors of 2, and 38! has

$\lfloor 38/2 \rfloor + \lfloor 38/4 \rfloor + \lfloor 38/8 \rfloor + \lfloor 38/16 \rfloor + \lfloor 38/32 \rfloor = 19+9+4+2+1 = 35$ factors of 2. Thus, the expression has $47 - 10 - 35 = \boxed{2}$ factors of 2.

Similarly, we determine how many factors of 5 divide the given number. 50! has $\lfloor 50/5 \rfloor + \lfloor 50/25 \rfloor = 10 + 2 = 12$ factors of 5, 12! has $\lfloor 12/5 \rfloor = 2$ factors of 5, and 38! has $\lfloor 38/5 \rfloor + \lfloor 38/25 \rfloor = 7 + 1 = 8$. Thus, our number has $12 - 2 - 8 = \boxed{2}$ factors of 5.

Our number has two factors of each of 2 and 5, so it has two factors of 10. Thus, our number has $\boxed{2}$ trailing zeros.

27. Notice that all the time intervals are multiples of ten; Thus, instead of doing 20, 30, and 50 minute intervals across 300 minutes, we can do 2, 3, and 5 unit intervals across 30 units. In other words, you want to find the number of times a unit is NOT a multiple of 2, 3, or 5, and then subtract from 31 (as it includes 0) to find the number of time Edwin wakes up. Therefore, you get $31 - \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{4}{5} * 30 = 31 - 8 = \boxed{23}$ times.

28. There are 10 possible points where Kelvin can take his superhop such that he can end up at BCA, all points where $x \in [2, 3]$ *and* $y \in [-1, 3]$. For each point, the number of paths through it can be written as $a \times b$, where a is the number of paths getting there, and b is the number of paths from that point to BCA. The total number of paths is $1 \cdot 1 + 2 \cdot 1 + 3 \cdot 1 + 4 \cdot 1 + 5 \cdot 1 + 1 \cdot 5 + 1 \cdot 4 + 1 \cdot 3 + 1 \cdot 2 + 1 \cdot 1 = \boxed{30}$.

29. By Tangent to Angle, we see that $\angle DAC = \angle ABC = x$. Summing the angles in $\triangle DAB$, we have $2x + 32° + 74° = 180° \implies x = \boxed{37°}$.

30. First, note that by Vieta's Formulas, the product of the roots of a monic, even-degree polynomial is equal to its constant term. In addition, note that the constant term of a polynomial is equal to the value of the polynomial at $x = 0$ (since every term except the constant term becomes 0). The monic polynomial $P(Q(x))$ has degree $3 \cdot 2018 = 6054$, which is even, hence the product of its roots is equal to its constant term, which is $P(Q(0))$. We may evaluate $P(Q(0))$ as $P(Q(0)) = P(2) = 2^{2018} + 2^{2017} + \cdots + 2^2 + 2^1 + 1$. This is a geometric series which simplifies to $2^{2019} - 1$. The smallest power of 2 greater than $2^{2019} - 1$ is 2^{2019}, so the answer is $y = \boxed{2019}$.

31. Denote R_n to be the number of ways to tile a 2 by n rectangle. There is 2 cases for how a 2 by n rectangle can be tiled. One case is that R_n rectangle ends with one vertical domino which gives R_{n-1}. The next case is with 2 horizontal dominoes at the end which gives R_{n-2}. This gives the recursive relationship $R_n = R_{n-1} + R_{n-2}$. Note that $R_1 = 1$ and $R_2 = 2$. We can solve for the rest of the terms. $R_3 = 3, R_4 = 5, R_5 = 8, R_6 = 13, R_7 = 21, R_8 = 34, R_9 = 55, R_{10} = 89$ (note this is the Fibonacci sequence). We get the answer $\boxed{89}$.

32. Let point E denote the intersection of the diagonals AC and BD. Furthermore, let points F and G denote the midpoints of AB and CD, respectively. It can be shown that the midpoint of the hypotenuse of a right triangle is equidistant from each of the vertices of the triangle. Thus, $FE = \frac{63}{2}$ and $GE = \frac{149}{2}$. Using similar triangles, one can show that points F, E, and G lie on the same line in that order; therefore, $FG = \boxed{106}$.

33. We use complementary counting by counting the number of labellings such that there is no pair of opposite faces whose faces sum to 7. WLOG, we can label any face with "1." Then, its opposite face must not be 6, but can be any number in the set $\{2, 3, 4, 5\}$, so WLOG, let label it "2." There are four faces left to be labelled. Note that we can choose any of these faces to be labelled "6" since we can freely rotate the cube about the axis through the center of the cube containing the centers of the faces with "1" and "2". We are left with three faces to label. Two of them are opposite each other, but we have one remaining pair summing to 7 (3 and 4), so both of them cannot be placed among these faces. Thus, the only remaining element, "5," must be placed in one of these two faces. Finally, we can choose which of the remaining two faces to put the "3" and "4". In this construction, we had 4 choices when we placed the number opposite "1,", 2 choices when we placed the "5," and 2 choices to place the "3" and "4," for a total of $4 \cdot 2 \cdot 2 = 16$ options.
Recall that we're counting the things that we don't want. To count the total number of labelings, we place the "1" on any face of the cube. Then, we have 5 choices for the number opposite "1." Finally, there are 4! choices for the remaining four numbers, but to account for the rotation of the cube, we divide by 4, which yields $3! = 6$ choices. Thus, there is a total of $5 \cdot 6 = 30$ labelings.
Hence, our answer is $30 - 16 = \boxed{14}$.

34. Label each square with the number of ways in which it is possible to reach that square. The top row contains one 1, and for rows 2 through 4, each of the numbers in the row is equal to the product of the number in the previous row and the number of squares in the previous row. (This is because one can choose to move down on any cell in the row.) For rows 5 and 6, each of the numbers in the row equals the product of the number in the previous row and the number of squares in the row itself (since one can only move down onto one of the squares in the row). Therefore, there are $1 \times 2 \times 4 \times 6 \times 4 \times 2 = \boxed{384}$ ways to get to the bottom right square.

35. We recognize the powers of three in the problem, and rewrite the given polynomial as $x^9 + 3^5x^3 + 3^6$. As the first and last terms are both cubes, we try to "complete the cube". Lo and behold, $(x^3+9)^3 = x^9 + 27x^6 + 243x^3 + 729$. Therefore, our polynomial is simply $(x^3+9)^3 - 27x^6$. This expression is a difference of cubes, which factors as $(x^3 - 3x^2 + 9)q(x)$, for some polynomial $q(x)$ of degree 6. Hence, the answer is $\boxed{x^3 - 3x^2 + 9}$.

36. Suppose that the number t has period 2. Therefore, it satisfies $P(P(t)) = t$, or, equivalently, $P(P(t)) - t = 0$. Moreover, it does not satisfy $P(t) - t = 0$, otherwise, it would have period 1. Thus, t satisfies $\frac{P(P(t))-t}{P(t)-t} = 0$. Substituting the expression for P and expanding yields $t^2 + 3t + 3$, whence by Vieta's Formulas, it follows that the answer is $\boxed{-3}$.

37. Let Andrew's and Simon's initial starting positions be A and S, respectively. Notice that the described circle is also the locus of all points P such that $\frac{AP}{SP}$ is some constant k, which is also the ratio of Andrew's to Simon's speeds. Consider the points on the locus which are also line $\overleftrightarrow{AS}$. There are two such points, one on segment $\overline{AS}$ and one not on ray $\overrightarrow{SA}$. Subsequently, the segment connecting these two points must be the diameter of the circle (if the segment were a chord, then $\overleftrightarrow{AS}$ wouldn't be a line of symmetry, which doesn't make intuitive sense). We can compute that given a ratio k, the radius of the circle is $r = \frac{24}{k+1} + \frac{24}{k-1}$ ($\frac{48}{k+1}$ is the distance from S to the point on segment $\overline{AS}$ and $\frac{48}{k-1}$ is the distance from S to the point not on ray $\overrightarrow{SA}$). When Andrew runs at 20 miles per hour and Simon runs at 12, $k = \frac{20}{12} = \frac{5}{3} \longrightarrow r = 90$. Setting $r = 45 - 13 = 32$ in the original equation, we get that $k = 2$ (an extraneous negative solution for k is also given). Thus, $\frac{20}{x} = k = 2 \longrightarrow x = \boxed{10}$ miles per hour.

38. We make two observations:

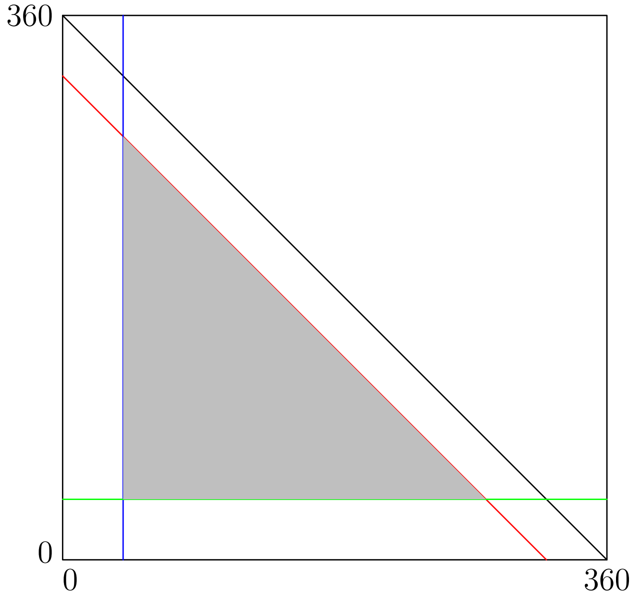
- After each ogre makes a move, the number of pieces on the board increases by 1.

- There is initially 1 piece, and the game terminates when there are 50 pieces on the board.

Thus 49 cuts must be made regardless of strategy, and since Shrek goes first, we need $49 \equiv 1 \mod n$. Thus $n \mid 48$ and there are $\tau(48) = \tau(2^4 3^1) = \boxed{10}$ values of n that work.

39. Call the triangle $\triangle ABC$, and WLOG fix point A arbitrarily on the circle. Then let x denote the measure of arc $\widehat{AB}$ and y denote the measure of arc $\widehat{BC}$. We know that arc $\widehat{AC}$ has to be positive, so $x + y < 360$. Moreover, any choice of x and y that satisfies $x + y < 360$ defines a unique triangle (the black line below). Therefore, the total area of valid triangles is the area of a 360 by 360 triangle isosceles right triangle. To find the desired region, we instead calculate the probability that the triangle has all angles greater than θ. In order for angle A to be greater than θ, arc $\widehat{BC}$ has to be greater than 2θ by the Inscribed Angle Theorem. Thus, $y > 2\theta$ (green line below). Similarly, we draw the lines $x > 2\theta$ and $360 - x - y > 2\theta$ (blue and red lines, respectively). The lower right intersection point of the red line and green line is at $(360 - 4\theta, 2\theta)$. Therefore, the triangle defined by these three lines has legs of length $360 - 6\theta$. Remember that this is the complement of what we actually want to count, so its area should be $1 - \frac{5}{9} = \frac{4}{9}$ of the total area. Thus, we want

$$\frac{\frac{(360^\circ - 6\theta)^2}{2}}{\frac{(360^\circ)^2}{2}} = \frac{4}{9} \implies \theta = \boxed{20^\circ}.$$



40. We claim that there are 6^5 solutions if $\gcd(a_1, \cdots, a_6) = 1$, and more solutions otherwise.

First, consider the case where $\gcd(a_1, \cdots, a_6) = 1$.

Case 1: There is an a_i such that $a_i = 1$ or $a_i = 5$.

Then we can assign arbitrary values to the other a_j ($j \neq i$) in 6^5 ways, and there will always be a unique solution to the congruence by choosing a_i appropriately (as $1, 5$ are invertible $\mod 6$.)

Case 2: All a_i are divisible by $2, 3$, or both.

Let x, y be the number of the a_i divisible by 2 but not 3 and 3 but not 2, respectively. Clearly $x, y > 0$, since otherwise the gcd condition breaks. Then we need $2\sum_x a_i + 3\sum_y a_i \equiv 0 \mod 6$, so $\sum_x a_i \equiv 0 \mod 3$ and $\sum_y a_i \equiv 0 \mod 2$. A similar analysis as above yields $2 \cdot 6^{x-1}, 3 \cdot 6^{y-1}$ solutions to each congruence. Moreover, we may assign any value to the remaining $6 - x - y$ x_i, so the answer follows.

Now it is easy to see by dividing each a_i as well as the modulus by the gcd in the case that the gcd is not 1 that there are $6^6 / \gcd(a_1, \cdots, a_6)$ solutions.

Finally, we count the solutions to $\gcd(a_1, \cdots, a_6) = 1$ by counting the complement: $6^6 - 3^6 - 2^6 + 1 = (2^6 - 1)(3^6 - 1) = 45864$. Dividing by 6^6 yields $\boxed{\frac{637}{648}}$.