Joe Holbrook Memorial Math Competition

7th Grade

October 13, 2019

General Rules

- You will have **75 minutes** to solve **40 questions**. Your score is the number of correct answers.
- Only answers recorded on the answer sheet will be graded.
- This is an individual test. Anyone caught communicating with another student will be removed from the exam.
- Scores will be posted on the website. Please do not forget your ID number, as that will be the sole means of identification for the scores.
- You may not use the following aids:
  - Calculator or other computing device
  - Compass
  - Protractor
  - Ruler or straightedge

  In addition, you must use the scrap paper supplied by the proctors.

Other Notes

- Write legibly. If the graders cannot read your answer, you will be given no credit for that question.
- Fractions should be written in lowest terms. Please convert all mixed numbers into improper fractions.
- For constants such as e or π, do not approximate your answer: for example, if the answer to a question is 7π, then you should not write 22 or 21.99.
- You do not need to write units in your answers.
- Rationalize all denominators. In addition, numbers within a square root must be squarefree, e.g. $\sqrt{63}$ should be written as $3\sqrt{7}$.
- Ties will be broken by the number of correct responses to questions 31 through 40. Further ties will be broken by the number of correct responses in the last five questions.

1. Susan has a potato farm that generates an average of 119 pounds of potatoes per week. On average, how many pounds of potatoes are generated per day?

2. Express $\dfrac{23}{4}$ in decimal form.

3. If $2 \cdot x + 0 \cdot x + 1 \cdot x + 9 \cdot x = 24$, what is x?

4. Compute $\dfrac{2017 + 2018 + 2019}{6}$.

5. In the land of BCAmerica, numbers work differently. For the set of five numbers $\{1,2,3,4,5\}$, the following inequalities are true: $3 < 4$, $1 < 2$, $4 < 1$, and $5 < 4$. Which number in the set is the median?

6. Expressed as a decimal, what is 1009 added to $\dfrac{1}{2019}$ of the answer to this question?

7. The sum of 3 consecutive integers is 15. What is the product of the three integers?

8. When given a number, a machine adds 6 to the number, multiplies the result by 2, subtracts 4, and outputs the result. A number is entered into the machine, and the machine outputs 14. What number was entered into the machine?

9. Let $\langle n \rangle$ denote the sum of all positive divisors of n, excluding n itself. For example, $\langle 4 \rangle = 1 + 2 = 3$. What is $\langle\langle\langle 6 \rangle\rangle\rangle$?

10. Define the function $a \circ b$ to equal the quantity $a^2 - b^2$. Compute $7 \circ (3 \circ 2)$.

11. Calculate $2^{0^{1^9}} \times 2018 - 2017$.

12. Abhinav is running for president against Susan. Everyone voted for either Abhinav, Susan, or both. Abhinav received 2019 votes, Susan received 2018 votes, and a total of 2020 students voted. How many people voted for both candidates?

13. For how many integers $k < 100$ does the equation $k = x^2$ have an integer solution x?

14. Let a and b be positive integers such that $1^4 + 5^4 + 6^4 + a^4 + b^4 = 2019$. Find $a + b$.

15. Vfire must walk 48 miles to reach the cookie fairy. He walks at 6 miles per hour when he is not carrying anything. However, he is forced to carry 120 pounds of cookies, and every 30 pounds of cookies makes him 1 mile per hour slower. How many fewer hours would it take Vfire to walk to the cookie fairy if he was not carrying anything?

16. If $2^x = 25$, what is $2^{\frac{x}{2}+3}$?

17. Find the sum of the coefficients of the polynomial $(2x + 1)^3$.

18. Daniel, Derek, and Sameer are canoeing during a thunderstorm, and their canoe capsizes 600 feet from the shore. Daniel drifts towards the shore at a rate of 20 feet per minute, Sameer swims at a rate of 60 feet per minute, and Derek swims at a rate of 75 feet per minute. For how many minutes are there exactly 2 people in the water?

19. What is the remainder when the product of the primes less than 100 is divided by 20?

20. Eric the human, his chickens, and his pigs live on the Ming Farm. Given that there are 21 heads and 52 feet on the farm, how many chickens are on the farm?

21. One day, Charles and his younger brother Eric realized that their ages were both prime numbers. Charles noted that the difference between their ages was 15. Eric then summed the cubes of their ages and got 4921. What was Eric's age?

22. If a and b are integers such that $a^b = 2^{14}$, what is the minimum possible value of $a + b$?

23. Points M, A, R, V, I, N, Y, and U lie in the plane such that
$$MA + AR + RV + VI + IN + NY + YU + UM = 24.$$
Given that $MARVINYU$ is a convex polygon with vertices in that order, find the maximum possible area of polygon $MARVINYU$.

24. There exist three digits A, B, and C such that both the numbers $x = 2591A0B$ and $y = 10242ABC$ are multiples of 9. What is C? (Here, A, B, and C represent digits in the decimal representations of x and y.)

25. If $(x + y)^2 = 37$ and $xy = 3$, what is $|x - y|$?

26. In the perfect world of Geometria, hamsters are perfect cylinders. When it gets cold, hamsters conserve heat by increasing their radii and decreasing their length, all while maintaining a constant volume. During summer, David's hamster has radius r and length $2r$. During winter, his hamster's radius increases by one and its length decreases by one. What is r?

27. Simon and Doug are enemies and refuse to sit together. On the other hand, Doug and Sumner are best friends and insist on sitting next to each other. How many distinct ways are there to seat Simon, Doug, Sumner, and Andy around a circular table with four seats? Two arrangements are considered indistinguishable if one can be obtained from the other by a rotation.

28. JazzyZ graphs the parabolas $y = x^2$ and $y = -x^2 + 4$ on the coordinate plane. Let V_1 and V_2 be the vertices of the former and latter parabolas, respectively, and let A and B be the intersection points of the two parabolas. Find the perimeter of quadrilateral AV_1BV_2.

29. Andy's standard 12-hour analog clock reads 1:30 currently. The lengths of the minute and hour hands are 15 units and 8 units, respectively. After how many minutes will the distance between the tips of the clock hands equal 17 units? (The tip of a clock hand refers to the point at the greatest radial distance from the center of the clock.)

30. How many non-degenerate isosceles triangles with integer side lengths have perimeter 60? (An equilateral triangle is considered to be isosceles.)

31. Let $A = 3 + \cfrac{1}{3 + \frac{1}{3 + \frac{1}{3 + \dots}}}$ and $B = \sqrt{3 + \sqrt{3 + \sqrt{3 + \dots}}}$. Evaluate $A - B$.

32. Find the real value of x which satisfies
$$20002x^3 + 10001x + 90009 = 20192019.$$

33. In quadrilateral $ABCD$, $\angle ABC = \angle CDA = 90^\circ$. Furthermore, $AB = 3, BC = 4$, and $CD = 1$. What is the area of quadrilateral $ABCD$?

34. Define a positive integer to be *Susanian* if it satisfies the following two properties: the sum of the digits is 9 and the number is divisible by 15. How many Susanian integers are there less than 500?

35. Compute the sum
$$\left\lfloor \frac{2^0}{3} \right\rfloor + \left\lfloor \frac{2^1}{3} \right\rfloor + \left\lfloor \frac{2^2}{3} \right\rfloor + \dots + \left\lfloor \frac{2^{11}}{3} \right\rfloor,$$
where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

36. Let $f(x)$ equal the area of the parallelogram with vertices $(0, 0), (x, 1), (1, x)$, and $(x+1, x+1)$. Compute the infinite sum $\dfrac{1}{f(2)} + \dfrac{1}{f(3)} + \dfrac{1}{f(4)} + \dots$.

37. Let $A_0B_0C_0$ be a triangle with $A_0B_0 = 13$, $B_0C_0 = 14$, $C_0A_0 = 15$. For $n > 0$, define the triangles $A_nB_nC_n$ such that A_n is the midpoint of $\overline{A_{n-1}B_{n-1}}$, B_n is the midpoint of $\overline{B_{n-1}C_{n-1}}$, and C_n is the midpoint of $\overline{C_{n-1}A_{n-1}}$. Find $A_0B_0 + A_1B_1 + A_2B_2 + \cdots$.

38. Bob has a deck of 50 cards, with each card labeled with a different number between 1 and 50, inclusive. He begins by drawing 3 cards at random and holding them in his hand. Each second, Bob looks at the cards in his hand, and if he sees one that is numbered less than 20, he replaces it with a card in the original deck that is numbered 20 or greater. He continues this procedure until all the cards in his hand are numbered 20 or greater. What is the probability that at least 2 of the cards in his final hand have the same tens digit?

39. Let $ABCD$ be an isosceles trapezoid with $AB \parallel CD$ with $AB = 5$, $CD = 15$, and $\angle ADC = 60^\circ$. If E is the intersection of $\overline{AC}$ and $\overline{BD}$ and O is the circumcenter of $\triangle ABC$, compute EO.

40. Jotaro and DIO are playing catch in the coordinate plane. Jotaro is currently holding the ball at $(0, 0)$. DIO is located at $(5, 0)$. There are two walls, one at the line $y = 1$ and one at the line $y = -1$. When the ball hits a wall, it bounces perfectly off the wall. If the ball must bounce exactly three times, what is the minimum distance it can travel from Jotaro to DIO?