Joe Holbrook Memorial Math Competition

7th Grade Solutions

October 13, 2019

1. There are 7 days in a week. Thus the number of pounds of potatoes generated per day is $119/7 = \boxed{17}$.
2. Simply perform division. We find $\boxed{5.75}$.
3. After simplification the equation becomes $12x = 24$. Solving for x, we find $x = \boxed{2}$.
4. Since 2017, 2018, and 2019 form an arithmetic progression, $2017 + 2018 + 2019 = 3 \cdot 2018$. Hence the answer is $\dfrac{3 \cdot 2018}{6} = \dfrac{2018}{2} = \boxed{1009}$.
5. It is given that 3 and 5 are both less than 4. Further, $4 < 1 < 2$. Thus, $\boxed{4}$ is the median.
6. Let x be the answer to this question. Then $x = 1009 + \dfrac{x}{2019}$, implying $x = \boxed{1009.5}$.
7. Let the three integers be $n-1, n, n+1$. Then their sum is $(n-1)+n+(n+1) = 3n = 15$, and hence the middle integer is $\frac{15}{3} = 5$. Since the three integers are 4, 5, and 6, their product is $4 \cdot 5 \cdot 6 = \boxed{120}$.
8. When a number x is entered into the machine, the machine adds 6, which gives $x+6$. Next, it multiplies by 2 and subtracts 4, which yields $2(x+6)-4$. We are given that $2(x+6)-4 = 14$. Solving this equation yields $x = \boxed{3}$.
9. Note that $\langle 6 \rangle = 1+2+3 = 6$, hence $\langle\langle 6 \rangle\rangle = \langle 6 \rangle = 6$, and $\langle\langle\langle 6 \rangle\rangle\rangle = \langle 6 \rangle = \boxed{6}$. (Fun fact: in number theory, we call n a *perfect number* if $\langle n \rangle = n$.)
10. We use PEMDAS. We have $3 \circ 2 = 3^2 - 2^2 = 9 - 4 = 5$. This leaves us with $7 \circ 5$, or $7^2 - 5^2 = 49 - 25 = \boxed{24}$.
11. To calculate the exponent tower, we first calculate $1^9 = 1$, then $0^{1^9} = 0^1 = 0$, and finally $2^{0^{1^9}} = 2^0 = 1$. Hence, the original expression equals $1 \times 2018 - 2017 = 2018 - 2017 = \boxed{1}$, our answer.
12. Let x be the number of people who voted twice. Let y be the number of people who voted once. We get the system of equations $x + y = 2020$ and $2x + y = 2019 + 2018$. Solving this yields $x = \boxed{2017}$.
13. Since x must be real, $k \geq 0$. Since x must be an integer, k must be a square. There are $\boxed{10}$ squares from 0 to 99 inclusive.
14. Isolating the sum $a^4 + b^4$, we find $a^4 + b^4 = 97$. Clearly, a and b must be less than 4. Testing $a = 3$, we find $b^4 = 97 - 3^4 = 16 \implies b = 2$. By symmetry the pair $(a, b) = (2, 3)$ satisfies the equation. Either way, the answer is $a + b = \boxed{5}$.
15. If Charles was not carrying any cookies, he would be able to walk to the cookie fairy in $\frac{48}{6} = 8$ hours. However, since he is carrying 120 pounds of cookies, he can only walk at $6 - \frac{120}{30} = 2$ miles per hour, so it will take him $\frac{48}{2} = 24$ hours to walk to the cookie fairy. This takes him an extra $24 - 8 = \boxed{16}$ hours.
16. Given $2^x = 25$, we raise both sides of the equation to the $\frac{1}{2}$ power, yielding $2^{\frac{x}{2}} = 5$. If we want $2^{\frac{x}{2}+3}$, then we should multiply both sides of the equation by $2^3 = 8$. Thus, $2^{\frac{x}{2}+3} = 5 \cdot 8 = \boxed{40}$.
17. *Method 1:* The sum of the coefficients of a polynomial $P(x)$ is simply $P(1)$. To see this, note that if $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, then

$$P(1) = a_n 1^n + a_{n-1} 1^{n-1} + \ldots + a_1 1 + a_0$$

$$= a_n + a_{n-1} + \ldots + a_1 + a_0,$$

precisely the sum of the coefficients of $P(x)$. Hence, the answer is $(2 \cdot 1 + 1)^3 = 3^3 = \boxed{27}$.

Method 2: Expanding the given polynomial, we find

$$(2x+1)^3 = (2x+1)^2(2x+1) = (4x^2+4x+1)(2x+1)$$
$$= 8x^3+12x^2+6x+1.$$

Thus the sum of the coefficients is $8+12+6+1=27$, as above.

18. Derek reaches the shore in $\frac{600}{75} = 8$ minutes, after which there are 2 people in the water. Sameer reaches the shore in $\frac{600}{60} = 10$ minutes, after which only Daniel is in the water. Thus, exactly 2 people are in the water for $10-8 = \boxed{2}$ minutes.

19. The product is divisible by $2 \cdot 5 = 10$, so the remainder is either 0 or 10. The remainder can't be 0, as the product is only divisible by one power of 2. Hence our answer is $\boxed{10}$.

20. Let c denote the number of chickens on the farm, and let p denote the number of pigs on the farm. Since there are 21 heads total, including Eric's, $c+p+1=21$. Also, since there are 52 feet total, including both of Eric's, $2c+4p+2=52$. Solving this system, we find the answer of $c = \boxed{15}$ chickens.

21. The first two sentences are enough to solve the problem. The difference between two prime numbers can only be odd if the smaller number is 2. Thus Eric was $\boxed{2}$ years old and Charles was 17.

22. Note that a can be negative. Hence, we take a to be the most negative it can be, so long b remains even. The least possible value of b is 2, which corresponds to the most negative value of $a=-128$, yielding a sum of $-128+2 = \boxed{-126}$.

23. The maximum possible area of a convex polygon with a fixed perimeter occurs when the polygon is equilateral. Thus, the maximum area of $MARVINYU$ occurs when the octagon is equilateral with side length $24/8 = 3$. To quickly compute the area of $MARVINYU$, denote $P_1 = MA \cap RV$, $P_2 = RV \cap IN$, $P_3 = IN \cap YU$, and $P_4 = YU \cap MA$. Then $P_1P_2P_3P_4$ is a square with side length $3+3\sqrt{2}$. The area of octagon $MARVINYU$ is equal to the area of $P_1P_2P_3P_4$ minus four times the area of the right isosceles triangle P_1RA. Thus, our answer is

$$(3+3\sqrt{2})^2 - 4 \cdot \frac{1}{2} \cdot \left(\frac{3\sqrt{2}}{2}\right)^2 = \boxed{18+18\sqrt{2}}.$$

24. Recall that if a number is divisible by 9, the sum of digits of the number must be divisible by 9. As 9 divides x, 9 must divide $2+5+9+1+A+0+B = 17+A+B$, implying that $A+B$ leaves a remainder of 1 when divided by 9. Similarly, as 9 divides y, 9 must divide $1+0+2+4+2+A+B+C = 9+A+B+C$, implying that $A+B+C$ leaves a remainder of 0 when divided by 9. It follows that C must leave a remainder of 8 when divided by 9. As C is a digit between 0 and 9, C must equal $\boxed{8}$, our answer.

25. Note that $|x-y| = \sqrt{(x-y)^2}$. Hence, to find $|x-y|$, it suffices to compute $(x-y)^2$. To finish, note that

$$\begin{aligned}(x-y)^2 &= x^2 - 2xy + y^2 \\ &= x^2 + 2xy + y^2 - 4xy \\ &= (x+y)^2 - 4xy \\ &= 37 - 3 \cdot 4 \\ &= 25,\end{aligned}$$

thus $|x-y| = \boxed{5}$.

26. The hamster's volume during summer is $\pi r^2 \cdot 2r = 2\pi r^3$. The hamster's volume during winter is $\pi(r+1)^2 \cdot (2r-1)$. As the hamster's volume is conserved, we have

$$2\pi r^3 = \pi(r+1)^2 \cdot (2r-1) \implies 2r^3 = (r+1)^2(2r-1)$$
$$\implies 2r^3 = 2r^3 + 3r^2 - 1 \implies 3r^2 = 1.$$

Solving, we find $r = \boxed{\frac{\sqrt{3}}{3}}$, as desired.

27. Since there are four seats around the table, Simon must sit across from Doug. Since Doug must sit next to Sumner, Sumner must either sit to the right or left of Doug. If Sumner sits to the right of Doug, then Andy must sit to the left of Doug, and similarly, if Sumner sits to the left of Doug, then Andy must sit to the right of Doug. These two cases are rotationally distinct, and hence there are $\boxed{2}$ ways to seat the four people.

28. Note that $V_1 = (0, 0)$ and $V_2 = (0, 4)$. To find the intersection points A, B, we solve $x^2 = 4 - x^2 \implies x^2 = 2 \implies x = \pm\sqrt{2}$. Then the intersections are $(-\sqrt{2}, 2)$ and $(\sqrt{2}, 2)$. Thus, A, B are $(\sqrt{2}, 2)$ and $(-\sqrt{2}, 2)$ in some order. Since these points form a rhombus by symmetry, we can simply find the length of one of the sides and multiply by 4. As the distance between $(\sqrt{2}, 2)$ and $(0, 0)$ is $\sqrt{2+4} = \sqrt{6}$, the answer is $\boxed{4\sqrt{6}}$.

29. For the distance between the clock hands to equal $17 = \sqrt{8^2 + 15^2}$, the smaller angle between the clock hands must equal 90 degrees. The earliest time the clock could form a 90 degree angle would be before 2 o'clock, some $m < 60$ minutes after 1 o'clock. In this case, the hour hand would be located at an angular position of $30 + \frac{m}{60} \cdot 30 = 30 + \frac{m}{2}$ degrees, while the minute hand would be located at an angular position of $360 \cdot \frac{m}{60} = 6m$ degrees. By inspection, the difference in these angular positions should be obtuse and hence equal 270, yielding $6m - \left(30 + \frac{m}{2}\right) = 270 \implies m = \frac{600}{11}$. Thus, it takes $\frac{600}{11} - 30 = \boxed{\frac{270}{11}}$ minutes for the hands to form 90 degrees.

   Fun fact: the (not necessarily acute or obtuse) angle between the hands of a clock at hour H, minute M is equal to $\frac{1}{2}|60H - 11M|$.

30. Let the isosceles triangle have sides a, a and b. As the perimeter is 60, $2a + b = 60$. Further, the triangle inequality implies $2a > b$. Hence, $60 = 2a + b > 2b \implies b < 30$. As $b = 60 - 2a$ must be even, b can be any even integer between 2 and 28, inclusive. Each of these 14 values of b yields a non-degenerate isosceles triangle, and hence, our answer is $\boxed{14}$.

31. We have $A = 3 + \frac{1}{A}$, hence $A^2 - 3A - 1 = 0$. Solving gives us $A = \frac{3 \pm \sqrt{13}}{2}$. Since $\sqrt{13} > 3$, we must choose the positive solution, implying $A = \frac{3 + \sqrt{13}}{2}$. Similarly, $B = \sqrt{3 + B}$, hence $B^2 - B - 3 = 0$ and $B = \frac{1 + \sqrt{13}}{2}$. Thus, $A - B = \boxed{1}$.

32. Notice that each of the coefficients of the polynomial, as well as the right hand side, are divisible by 10001. Dividing the equation by 10001, we are left with $2x^3 + x + 9 = 2019$. We recognize that using base 10 expansion, $2019 = 2 \cdot 10^3 + 1 \cdot 10 + 9$, and hence our answer is $x = \boxed{10}$.

33. By the Pythagorean Theorem on $\triangle ABC$, $AC = \sqrt{3^2 + 4^2} = 5$. By the Pythagorean Theorem on $\triangle CDA$, $DA = \sqrt{5^2 - 1^2} = 2\sqrt{6}$. Since $\triangle ABC$ and $\triangle CDA$ are right triangles, their areas are $\frac{AB \cdot BC}{2} = 6$ and $\frac{CD \cdot DA}{2} = \sqrt{6}$, respectively. Since the area of $ABCD$ equals the sum of the areas of $\triangle ABC$ and $\triangle CDA$, our answer is $\boxed{6 + \sqrt{6}}$.

34. Notice that if the sum of the digits of a number is 9, then it must be divisible by 9 by the divisibility rule for 9. Therefore, Susanian integers have the property that they are divisible by 9 and 15. This is equivalent to being divisible by their least common multiple which is 45. However, it is important to realize that not all numbers that are divisible by 9 have sum of digits 9 - it could have a sum of digits that is a multiple of 9. Therefore, we test all multiples of 45 less than 500 to see if they have sum of digit 9. These multiples are $45, 90, 135, 180, 225, 270, 315, 360, 405, 450, 495$. Notice that $\boxed{10}$ of these are Susanian but 495 has sum of digit 18 and is not.

35. Note that for all $k \geq 0$, 2^{2k} leaves a remainder of 1 when divided by 3, while 2^{2k+1} leaves a remainder of 2 when divided by 3. Thus, $\left\lfloor \frac{2^{2k}}{3} \right\rfloor = \frac{2^{2k} - 1}{3}$ and $\left\lfloor \frac{2^{2k+1}}{3} \right\rfloor = \frac{2^{2k+1} - 2}{3}$, implying that $\left\lfloor \frac{2^{2k}}{3} \right\rfloor + \left\lfloor \frac{2^{2k+1}}{3} \right\rfloor = \frac{2^{2k} + 2^{2k+1}}{3} - 1$ for all $k \geq 0$. Since the desired sum equals the sum of $\left\lfloor \frac{2^{2k}}{3} \right\rfloor + \left\lfloor \frac{2^{2k+1}}{3} \right\rfloor$ for $0 \leq k \leq 5$,

it follows that the desired sum equals

$$\left(\frac{2^0+2^1}{3}-1\right)+\left(\frac{2^2+2^3}{3}-1\right)+\ldots+\left(\frac{2^{10}+2^{11}}{3}-1\right)$$
$$=\frac{2^0+2^1+\ldots+2^{11}}{3}-6=\frac{2^{12}-1}{3}-6=\boxed{1359}.$$

36. The area of the parallelogram with vertices $(0,0),(x,1),(1,x)$, and $(x+1,x+1)$ is x^2-1. Thus,

$$\frac{1}{f(x)}=\frac{1}{x^2-1}=\frac{1}{2}\left(\frac{1}{x-1}-\frac{1}{x+1}\right).$$

Thus, the desired infinite sum equals

$$\frac{1}{2}\left(\left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\ldots\right).$$

This sum telescopes to $\frac{1}{2}\left(\frac{1}{1}+\frac{1}{2}\right)=\frac{1}{2}\cdot\frac{3}{2}=\boxed{\frac{3}{4}}$, which is the answer.

37. Note that $\triangle A_nB_nC_n \sim \triangle B_{n+1}C_{n+1}A_{n+1}$ with scale factor $\frac{1}{2}$. This means

$$\begin{aligned}A_0B_0+A_1B_1+A_2B_2+\cdots &= A_0B_0+\frac{C_0A_0}{2}+\frac{B_0C_0}{4}+\cdots\\ &=\left(A_0B_0+\frac{C_0A_0}{2}+\frac{B_0C_0}{4}\right)\left(1+\frac{1}{8}+\frac{1}{8^2}+\cdots\right)\\ &=\left(13+\frac{15}{2}+\frac{14}{4}\right)\cdot\frac{1}{1-1/8}\\ &=\boxed{\frac{192}{7}}.\end{aligned}$$

38. Note that Bob's 3 card selection process is equivalent to simply choosing 3 cards from 20 to 50 inclusive at random. We calculate the desired probability by complementary counting: in other words, we will find the probability that all 3 cards have different tens digits.

There are 4 possible tens digits: 2, 3, 4, and 5. There are 10 cards with tens digit 2, 10 with tens digit 3, and 10 with tens digit 4, while there is only one with tens digit 5. If none of the 3 cards have tens digit 5, then there must be one with tens digit 2, one with tens digit 3, and one with tens digit 4: this can occur in $10\cdot10\cdot10=10^3$ ways. There are 3 ways in which one of the cards can have tens digit 5, and each of these cases has $10\cdot10=10^2$ ways. As there are a total of $\binom{31}{3}$ selections of 3 cards, the complement of our desired probability is $\frac{10^3+3\cdot10^2}{\binom{31}{3}}=\frac{260}{899}$, and hence our answer is $1-\frac{260}{899}=\boxed{\frac{639}{899}}$.

39. Let X be the intersection of $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$ so that $\triangle XCD$ is equilateral.

Since $ABCD$ is an isosceles trapezoid, it has a cirumcircle, i.e. D lies on the circumcircle on $\triangle ABC$. Now, as $\angle AED$ is an exterior angle to isosceles triangle $\triangle ECD$, $\angle AED = 2\angle ACD$. But $2\angle ACD=\angle AOD$, implying $\angle AED=\angle AOD$, or equivalently, that quadrilateral $AEOD$ is cyclic. Finally, calculating the power of the point X with respect to circle $AEOD$, we find

$$XE\cdot XO = XA\cdot XD \implies XO=\frac{5\cdot 15}{15\sqrt{3}/4}=\frac{20\sqrt{3}}{3}\implies EO=\boxed{\frac{35\sqrt{3}}{12}}.$$

40. Instead of reflecting the ball once it hits a wall, we reflect the entire figure as shown below, noting that the distance the ball travels is the same in both situations.

Then the vertical distance traveled is 6 and the horizontal distance 5, yielding an answer of $\sqrt{5^2+6^2}=\boxed{\sqrt{61}}$ by the Pythagorean Theorem.