Joe Holbrook Memorial Math Competition

5th Grade Solutions

October 18, 2020

1. The answer is $4 * 25 + 3 * 10 + 2 * 5 + 1 * 1 = \boxed{141}$ cents.

2. We perform the addition/subtraction of $76 + 12 - 2 = \boxed{86}$.

3. From the fact that 4 scores and 7 years is equivalent to 87 years, we can see that a score is 20 years. Therefore, 2020 years is equivalent to $\frac{2020}{20} = \boxed{101}$ scores.

4. There are 14 choices for a consonant and 10 choices for a vowel, so $14 * 10 = \boxed{140}$.

5. Alex will take $\frac{3000}{50} = 60$ minutes, and Yoland will take $\frac{3000}{120} = 25$ minutes, so $60 - 25 = \boxed{35}$ minutes.

6. $2.5 \text{ hr} * \frac{60 \text{ min}}{1 \text{ hr}} = 150 \text{ min}$; $\frac{5 \text{ things}}{1 \text{ min}} * 150 \text{ min} = \boxed{750}$

7. The least common multiple of 10 $(2 \cdot 5)$, 15 $(3 \cdot 5)$, and 4 $(2 \cdot 2)$ is $2 \cdot 2 \cdot 3 \cdot 5 = \boxed{60}$.

8. The number of four letter words is $500 - (27 + 166 + 92) = 215$, so the probability of one being selected is $\frac{215}{500} = \frac{43}{100}$. So $43 + 100 = \boxed{143}$

9. The probability that Erik doesn't damage his computer at a single error message is $1 - \frac{2}{3} = \frac{1}{3}$. This happens 6 times in a row with probability $(\frac{1}{3})^6 = \frac{1}{729}$. $1 + 729 = \boxed{730}$

10. $\frac{9 \cdot 12}{2} = 54$ boxes can fit vertically, and $\frac{16 \cdot 12}{3} = 64$ boxes can fit horizontally. $54 \cdot 64 = \boxed{3456}$

11. Plugging in (for convenience) 0 degrees Gankine and 4 degrees Gankine, we have 4 degrees Farenheit and 50 degrees Farenheit respectively. Subtracting yields $\boxed{46}$.

12. If h is the change in height, since $V = l \cdot w \cdot h$, then $V = 24 \cdot 12 \cdot h$. This means that $864 = 288h$, so h = 3. Thus, the answer is $\boxed{3}$ inches.

13. There are $10 - 1 = 9$ students who own a pet. Of those $9 - 6 = 3$ of those students do not own a cat. Therefore, 3 out of the 7 dog owners do not own a cat, meaning that $7 - 3 = 4$ dog owners own a cat. Therefore, our answer is $\boxed{4}$.

14. There are two ways this can occur. Super Cool Jessica can role HT or TH. Each occur with probability $\frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}$. Therefore, we arrive at the answer of $\frac{4}{9}$ which yields $\boxed{13}$.

15. We set x as the number of days that Jess does not practice. Her goal for October can then be expressed as $60 - 3(31 - x) + 5x < 10$. This simplifies to $8x < 43$ and x can be at most $\boxed{5}$.

16. The 95 raised Albert's average for 5 quizzes up by 5 points, which means it must have been $5 \cdot 5 = 25$ points higher than his lowest grade previously. $95 - 25 = \boxed{70}$.

17. There are $135 \cdot \frac{2}{9} = 30$ grams of baking powder, $30 \cdot \frac{3}{1} = 90$ grams of sugar, $90 \cdot \frac{7}{2} = 315$ grams of flour, and $315 \cdot \frac{2}{5} = \boxed{126}$ grams of milk.

18. We just need to subtract the volume of the flesh from the volume of the whole watermelon. The volume of a sphere is $\frac{4}{3}\pi r^3$. The radius of the watermelon is $\frac{12}{2} = 6$, while the radius of the flesh is $6 - 1 = 5$.
$$\frac{4}{3}\pi(6^3) - \frac{4}{3}\pi(5^3) = \frac{364\pi}{3}$$
$x = \boxed{367}$.

19. Let x by Betsy's height, let a be table A's height, and let b by table B's height (all in inches). When she stands on table A, the total height of her plus the table is $x + a$, while table B is b inches tall. So, $x + a - b = 36$. Similarly, when she stands on table B, her total height is $x + b$, and table A's height is a inches. So, $a - (x + b) = 4$. Subtracting the second equation from the first, we get $2x = 32$, so Betsy's height is $\boxed{16}$.

20. For any given one of the 200 points, we can find exactly one square with that point as one of the vertices. Since square has 4 vertices, to avoid repeated counting, we divide by four. So there are $200/4 = \boxed{50}$ squares.

21. We look to maximize the number at every stage of picking the digits. The largest 2-digit prime number is 97, and conveniently, the largest 2-digit prime with a tens unit of 7 is 79. Therefore, we should begin with a 9, then a 7, then a 9, and repeat in this matter. Therefore, our answer is $\boxed{97979}$.

22. If n is divisible by 5, it must end with a 0 or 5. We see what happens if the hundreds digit is a 9. If the units digit is a 0, and the middle digit is x, then 9+x+0 must be divisible by 6. This happens when x=3, 9. Of these, 990 is the biggest integer we can make. If the units digit is a 5, and the middle digit is y, then 9+y+5 must be divisible by 6. This happens when y=4, making 945. 990 ¿ 945, so our answer is $\boxed{990}$.

23. Given that no student can sit next to another student, we can quickly list the 4 possible places for the 3 students to sit. Since the students are distinct, in each possible seating there are $3! = 6$ ways to order the students. Thus there are a total of $6 * 4 = \boxed{24}$ possible ways for the students to sit.

24. If he drops x balls, there are x! sequences possible. He drops somewhere between 1 and 4 balls inclusive. Therefore, our answer is $1! + 2! + 3! + 4! = \boxed{33}$.

25. We want to sum the first n numbers, but subtract out one. Therefore, we want $(n)(n+1)/2$ to be greater than 200, but by a quantity less than or equal to n. Consider taking $n = 20$. We see that the sum becomes 210, so $n = 20$. Furthermore, we conclude that $m = 10$ as $210 - 200 = 10$. In summary, $m + n = \boxed{30}$. Note that there is a unique solution to for this problem; that is there is only one pair of values n and m. Can you prove this?

26. This problem asks to find the sum of $1/2^1 + 1/2^2 + 1/2^3 \cdots + 1/2^8$. Either using the formula for a geometric series or with pattern detecting, one can evaluate this to be $\frac{255}{256}$. We conclude that the $p + q$ is $\boxed{511}$.

27. Notice $\triangle EFB \sim \triangle EDA$, so $\frac{EF}{ED} = \frac{BE}{AE} = \frac{1}{3}$. $ED = 3 * EF = 3 * 5 = \boxed{15}$.

28. We start with the second person and notice that he used the digit 3. This means that this person must be speaking in base five. Therefore, we know that $20_5 = 10$ people speak base $3_5 = 3$ and $11_5 = 6$ people speak both. To determine what base the first person speaks in we notice that we must have $6 = 20_?$ where $? = 3$ or 5. We see that ? must be 3. Knowing that the first person speaks base 3, we have that $101_3 = 10$ people speak base $12_3 = 5$. Hence, by PIE, there are $10 + 10 - 6 = \boxed{14}$ residents.

29. To find n, we must first minimize the number of digits. This means that we want to make our digits as big as possible so we do not need as many digit places to form the product. Clearly, we cannot do anything with 7 because it is prime and at least 5. $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = 2^4 \cdot 3^2 \cdot 5$, so we take the digit $3^2 = 9$ because 9 is the best we can do. 5 has to go by itself, and we are left with the possible digits of 2 and 8 or 4 and 4. For each of these cases, the resulting numbers are 25789 and 44579, so $n = 25789$, and the sum is $\boxed{31}$.

30. Let's say the expected number of mango pieces Alicia will eat is E. If the piece she eats is sour, then she will only be eating that piece. If the piece she eats is sweet, then she will eat $\frac{2}{5} \cdot E + 1$ pieces. This yields the equation, $\frac{3}{5} \cdot 1 + \frac{2}{5} \cdot (E + 1) = E$. Solving this equation, we get $E = \frac{5}{3}$ and the answer is $5 + 3 = \boxed{8}$.

31. Note that any odd number can be taken down to either 1 or 3 after repeatedly subtracting 4. If it ends in 1, simply multiply by 5. Otherwise, $\times 3, -4, -4, \times 5$ will make 5. So, any odd number is not *good*. Note that all these operations preserve parity for even numbers, ie they always stay even. This means that since 5 is odd, no even number can become 5. Then the first three *good* numbers greater than 100 are $102, 104, 106$, which sum to $\boxed{312}$.

32. Consider the angle between the hour hand and the minute hand. Every minute, the minute hand moves forward $\frac{1}{60} \cdot 360 = 6$ degrees, and the hour hand moves forward $\frac{1}{60} \cdot 30 = \frac{1}{2}$ degrees. In other words, every minute the angle measure between the hands shrinks by $6 - \frac{1}{2} = \frac{11}{2}$ degrees.
When the mantis is $5\sqrt{3}$ cm away from the fly, the hands and the line connecting the mantis and fly form a 30-60-90 triangle, and the angle between the hands is 60 degrees. Since it is 3 o'clock right now, the current angle measure is 90 degrees. If m represents the number of minutes before the angle measure is 60 degrees, then

$$60 = 90 - \frac{11}{2}m$$

$$m = \frac{60}{11}$$

Rounded to the nearest minute, it will take $\boxed{5}$ minutes for the mantis to grab the fly.

33. Note that the 2 by 2 grid condition implies that the four primes $(2, 3, 5, 7)$ are in the four corners. The only pairs of numbers that multiply to a square are any of $1, 4$, and 9 as well as 2 and 8. This means since the remaining numbers go in a cross shape, $1, 4$ and 9 have to be on the outside, and the middle number is either 6 or 8. If it is 8, then there are $24 \cdot 24$ ways (since the 4 primes as well as the 4 non-primes can both be permuted). If it is 6, then it is only $24 \cdot 12$ ways, since the 8 can't be next to the 2. It follows that the answer is $24 \cdot 36 = \boxed{864}$.

34. We want to maximize the number of sections we get with each cut. To do this, notice that each cut should intersect all previous cuts, but not at preexisting intersection points. So the nth cut would intersect $n-1$ chords, creating n new sections. We quickly realize that with n cuts, the maximum number of sections is $1 + T_n$, where T_n is the nth triangular number, or $1 + \frac{n(n+1)}{2}$. Since we have 138 people (including the captain), $1 + \frac{n(n+1)}{2} \geq 138$. The smallest value of n is $\boxed{17}$.

35. Let the cube's vertices be denoted $ABCDEFGH$, where $ABCD$ and $EFGH$ are two squares that form two of the parallel faces of the cube. The four edges connecting these two squares are AE, BF, CG, and DH. Let A be the starting point, and G be the endpoint. First of all, note that in in the no-fire case, the shortest path is of length $\sqrt{5}$, passing through A, the midpoint of BC and G. In fact, there are two other paths that work as well, similar to the first path, but instead passing through the midpoint of HD and EF respectively. Note that each of these paths only passes through two of the faces of the cube, and for each path, these two faces are distinct. It follows that even if any two faces are set on fire, by the pigeonhole principle, at least one of the 3 paths will remain safe from fire, hence $\sqrt{5}$ is always possible, meaning the expected shortest path is $\sqrt{5}$. $8\sqrt{5} = \sqrt{320}$, which is really close to $\sqrt{324} = 18$, so our answer is $\boxed{18}$.

36. Any element of S must have the exponent of 11 in its prime factorization be greater than or equal to 1. For primes 2, 3, 5, and 7, they must be greater than or equal to zero. So, using the formula for the sum of an infinite geometric series, the answer is equal to $(1 + \frac{1}{2^1} + \frac{1}{2^2} + \dots)(1 + \frac{1}{3^1} + \frac{1}{3^2} + \dots)(1 + \frac{1}{5^1} + \frac{1}{5^2} + \dots)(1 + \frac{1}{7^1} + \frac{1}{7^2} + \dots)(\frac{1}{11^1} + \frac{1}{11^2} + \dots) = 2 \cdot \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{7}{6} \cdot \frac{1}{10} = \frac{7}{16}$. The final answer is $\boxed{23}$.

37. Since $AB = AC = AD$, we can draw a semicircle with A as the center and AB as the radius. Let B' be the point on the circle that's on the diameter with A and B. Then $BCDB'$ is an isosceles trapezoid, meaning $B'D = 4$. $B'DB$ is a right triangle because one of its sides is the diameter of the circle and all three vertices are on the circle, so $BD = \sqrt{B'B^2 - DB'^2} = \sqrt{10^2 - 4^2} = 2\sqrt{21}$ and $a + b = \boxed{23}$.

38. The probability that a positive integer is divisible by 7^k is equal to $\frac{1}{7^k}$. The probability that the prime factorization of an integer has exactly k number of 7's is equal to the probability that it is divisible by 7^k, but not 7^{k+1}. Therefore, the probability is equal to $\frac{1}{7^k} - \frac{1}{7^{k+1}} = \frac{6}{7^{k+1}}$. Therefore, the probability in the

question can be written as $\frac{6}{7} + \frac{6}{7^4} + \ldots$. Using the formula for a geometric series, our answer is $\frac{\frac{6}{7}}{1 - \frac{1}{343}}$ $= \frac{49}{57}$. Therefore, our answer is $\boxed{106}$.

39. Let $AB = CD = 20 = x$ and $BC = DA = 37 = y$. For short, let P be the length of the perimeter of the rectangle. We consider 4 cases based on the location of point E. **Case 1:** Point E lies on side AB. The probability of this case occurring is $\frac{x}{P}$. The only way for the two segments to intersect is if F lies between A and E. The expected position of point E is halfway between A and B. Thus, the overall probability is equal to the probability of the segments intersecting when E lies in this 'average' position. In this position, the probability that F lies between A and E is $\frac{x}{2P}$. The final probability in this case is $\frac{x^2}{2P^2}$. **Case 2:** Point E lies on side BC. The probability of this occurring is $\frac{y}{P}$. For the two segments to intersect, F must be either between B and E or on side AB. When E lies in the expected position, the probability of the intersection is $\frac{x + y/2}{P} = \frac{2x + y}{2P}$. Thus, the final probability in this is $\frac{2xy + y^2}{2P^2}$. **Case 3:** Point E lies on side CD. The probability of this occurring is $\frac{x}{P}$. For the two segments to intersect, F must lie either between E and D or on side AD. In the expected position for E, the probability of this occurring is $\frac{y + x/2}{P} = \frac{x + 2y}{2P}$. The final probability in this case is then $\frac{x^2 + 2xy}{2P^2}$. **Case 4:** Point E lies on side DA. The probability of this happening is $\frac{y}{P}$. For the two segments to intersect, F must lie between A and E. In the expected position for E, the probability of this happening is $y2P$. So, the final probability in this case is $\frac{y^2}{2P^2}$. The answer is

$$\frac{2x^2 + 2y^2 + 4xy}{2P^2} = \frac{2(x+y)^2}{2P^2} = \frac{1}{4}.$$

Thus the final answer is $\boxed{5}$.

40. We can establish a 1-1 correspondence between the number of ways for Justin to win and the number of ways to choose 4 rows and 4 columns in a six by six grid. To see why this is, we can consider a simpler case. Suppose that instead of choosing 4 spots for 4 non distinct balls, we are choosing 2 spots for 2 non distinct balls. Given any combination of 2 rows and 2 columns that intersect at 4 squares, there exists a unique orientation of 2 balls in those 4 squares satisfying the condition that no balls lies on the right of and above another ball and that there are no balls in the same row and column. Therefore, the number of ways for Justin to win is $\binom{6}{4} \cdot \binom{6}{4}$, and the number of ways to choose 4 balls out of $6^2 = 36$ spots is $\binom{36}{4}$. Hence, the desired probability is $\frac{\binom{6}{4} \cdot \binom{6}{4}}{\binom{36}{4}}$, which simplifies to $\frac{5}{7 \cdot 17 \cdot 11}$. $m = 5, n = 7 \cdot 17 \cdot 11 \implies m + \frac{n}{11} = 5 + \frac{7 \cdot 17 \cdot 11}{11} = 5 + 119 = \boxed{124}$.