Joe Holbrook Memorial Math Competition

7th Grade Solutions

October 18, 2020

1. Note that one of the 5^2 terms in $\frac{5^2 \times 5^2}{5^2}$ can be reduced, meaning that the first part of our expression can be simplified to $5^2 = 25$. For the second part, we can break the fraction $\frac{5^2+5^2}{5^2}$ into $\frac{5^2}{5^2} + \frac{5^2}{5^2} = 1 + 1 = 2$. Therefore, our final sum is $25 + 2 = \boxed{27}$

2. $35 + 2 + 4.2 \cdot 60 + 2.5 \cdot 24 \cdot 60 + 1.6 \cdot 60 + 4 = \boxed{3989}$

3. 4:37pm to 5:52 pm is $15 + 60 = 75$ minutes. Adding the additional 87 minutes, Frank will have $75 + 87 = \boxed{162}$ total minutes to trick-or treat.

4. The width is $1920 \div 6 = 320$ while the height is $1080 \div 6 = 180$. The sum is $\boxed{500}$.

5. Let t be the number of turtles and d be the number of ducks. Thus we have the equations $t + d = 17$ and $4t + 2d = 60$ from the respective pieces of information we were given. Multiplying the first equation by 2 and subtracting from the second, we get $2t = 26$ so $\boxed{t = 13}$

6. Since Andy has 6 left, he must have gotten $6 \times 2 = 12$ from Bettie. This is one-third of Bettie's original amount, or $12 \times 3 = 36$. After giving 12 to Andy, Bettie has 24 left. After her brother steals half, Bettie has 12 left. Eating two, Bettie now has $\boxed{10}$ chocolates.

7. The formula for a triangle is $\frac{bh}{2}$ where b is the length of the base and h is the length of the height. The area of A is then $\frac{12 * 9}{2} = 54$. Letting the height to the side of length 18 of Triangle B be x, the area of Triangle B is $\frac{18 * b}{2} = 9b$. Since the areas of the two triangles are equal, $9b = 54$ and $\boxed{b = 6}$.

8. We try to see if there is any year beginning with the digit 2 that will satisfy the conditions. Then, the years can only be in the forms of 2__2, 2_2_, or 22__. The first year bigger than 2020 in each form is 2112, 2121, and 2200. Of these, $\boxed{2112}$ is the least.

9. 61 and 67 are prime numbers and only have 2 factors. 62 $(2 \cdot 31)$ and 65 $(5 \cdot 13)$ only have two prime factors each and thus 4 total factors. 63, 64, and 66 satisfy Kevin's conditions as they have 6, 7, and 8 factors respectively. Thus the answer is $\boxed{4}$.

10. For each district denoted by (a, b) considering evaluating the fraction b/a. This is the how many votes a dollar can buy. b/a for districts 1 to 5 is 1, 4/3, 2, 5, 2 respectively. So, ideally Vivian would buy votes from the districts with highest b/a. Conveniently, she can buy the districts 3, 4, and 5 for \$19 dollars and obtain $\boxed{50}$ votes.

11. Let the length of the bathroom before it is expanded be l, and the width pre-expansion be w. We know that l and w are integers, and $(l + 2)(w + 3) = 45$. Listed below are all possible values for $l + 2$ and $w + 3$, along with the corresponding values of l and w:

$$l + 2 = 3, w + 3 = 15 \Rightarrow l = 1, w = 12, lw = 12$$

$$l + 2 = 5, w + 3 = 9 \Rightarrow l = 3, w = 6, lw = 18$$

$$l + 2 = 9, w + 3 = 5 \Rightarrow l = 7, w = 2, lw = 14$$

Clearly, the largest possible area pre-expansion is $\boxed{18}$."

12. After n hours of practice, Yul's score is $50n(n + 1) + 1500$. The first n that makes this bigger than $15,000$ is $n = \boxed{16}$

13. The prime factorizations are as follows:

$$2020 = 2^2 \cdot 5 \cdot 101$$

$$606 = 2 \cdot 3 \cdot 101$$

$$2222 = 2 \cdot 11 \cdot 101$$

The greatest common divisor is $2 * 101 = 202$, which has divisors of 1, 2, 101, and 202. The sum is therefore $\boxed{306}$.

14. Isaac's first hint tells us that his parents' ages have either 3 or 4 factors. If you play around with the prime factorizations of numbers, you will find that the only numbers with 3 factors are the squares of primes, and the only numbers with 4 factors are the products of 2 distinct primes.
Isaac's second hint restricts his parents' ages to between 20 and 50. Thus, Isaac's parents can only be 21, 22, 25, 26, 33, 34, 35, 38, 39, 46, 49. Isaac also tells us his parents' ages end in 9, so they can only be 39 or 49. Since Isaac's dad is older than his mom, we know his dad is 49 and his mom is 39. $39 + 49 = \boxed{88}$.

15. The easiest way to go about this is to draw a three-way venn diagram. Note that because a person's camera being on implies they are dressed up, the area of the "camera on" circle outside of the "dressed up" circle is empty. The intersection of all three circles is 2, and all areas can be deduced from here. There are 5 people with only mic on, 10 people only dressed up, 3 people with mic on and dressed up but no camera, 22 people with camera on and dressed up but no mic, and 2 with all three, for a total of 42 people with at least one of those things. $80 - 42 = \boxed{38}$

16. In order to maximize the amount of time it takes Emma to split the square into 18 cells, we wish to maximize two things: The amount of time it takes her to draw a line, and the number of lines she draws to split the square into 18 cells. When she splits the square into 25 cells, she either splits it into a grid with 5 rows and 5 columns, or 1 row and 25 columns (or vice versa). The first way requires 8 lines, and the second way requires 24 lines. Since we want to maximize the amount of time it takes her to draw one line, we assume that she only drew 8 lines in 48 seconds. Thus, it takes her a maximum of 6 seconds to draw one line. Now, when splitting the square into 18 cells, she can either use 1 by 18, 2 by 9, or 3 by 6. She can draw the most number of lines by using 1 by 18, and thus drawing 17 lines. So, if she takes a maximum of 6 seconds per line, and she draws a maximum of 17 lines, then the maximum number of time it can take her to split the square into 18 cells is $6*17 = \boxed{102}$.

17. There are 3! ways for A, B, and C to choose their cabins and there are $\binom{6}{3}\binom{3}{2}\binom{1}{1}$ ways for the other 6 friends to choose their cabins. Thus, there are $6 \cdot 20 \cdot 3 = \boxed{360}$ ways.

18. We can multiply both sides of the congruence $8^{-1} \equiv 8^2 \pmod{m}$ by 8:

$$\underbrace{8 \cdot 8^{-1}}_{1} \equiv \underbrace{8 \cdot 8^2}_{8^3} \pmod{m}.$$

Thus $8^3 - 1 = 511$ is a multiple of m. We know that m has two digits. The only two-digit positive divisor of 511 is 73, so $m = \boxed{43}$.

19. Let a be the proportion of the house that Albert can finish in one day. Let b, c, d be the same for Bertha, Clyde, and Dale respectively. We know, based on the problem, that

$$(1)\text{: } 2a + b = \frac{1}{4}$$

$$(2)\text{: } b + 2c = \frac{1}{8}$$

$$(3)\text{: } a + b + c + d = \frac{1}{2}$$

If we add equations 1 and 2, we get

$$2a + 2b + 2c = \frac{3}{8} \Rightarrow a + b + c = \frac{3}{16}$$

Subtracting this expression from equation 3, we have

$$d = \frac{5}{16}$$

This means Dale finishes $\frac{5}{16}$ of the house per day, so he will take $\frac{16}{5}$ days to finish. This rounds down to $\boxed{3}$ days."

20. WLOG, assume that $AD = 3$ and $DC = 7$. By the angle bisector theorem, $\frac{BC}{AB} = \frac{7}{3}$. If $BC = 7$ and $AB = 3$, the triangle is degenerate, and if $BC = 14$ and $AB = 6$, the triangle inequality holds. Hence, the answer is $14 + 10 + 6 = \boxed{30}$.

21. Notice that no one can get 1 apple, since that means no one else can get more than 2 apples. However, this is not possible since there are 15 apples. So, the minimum number of apples that anyone gets is either 2 or 3 (it cannot be more, since $15/5 = 3$). If the minimum is 3, then everyone gets 3 apples, which can happen in one way. If the minimum is 2, then no one gets more than 4 apples, so there 2 possibilities here: three people get 3 apples, one gets 2 apples, and one gets 4 apples, or 1 gets 3 apples, 2 get 2 apples, and 2 get 4 apples. The first case can happen in $5 \cdot 4 = 20$ ways, since there are 5 choices for who gets 4, and 4 choices for who gets 3. Similarly, in the second case, there are $\binom{5}{2} \cdot \binom{3}{2} = 30$ ways. The final answer is $1 + 20 + 30 = \boxed{51}$.

22. In any given game, there can be 8, 7, or 5 twirls. The first time where 5 consecutive numbers can be made is from 12 to 16. $12 = 5 + 7$, $13 = 5 + 8$, $14 = 7 + 7$, $15 = 7 + 8$, $16 = 8 + 8$, and multiples of 5 can be added to these to make any number that's greater than equal to 14 so $\boxed{11}$ is the greatest number that is not possible.

23. First, we factor 10! to become $2^8 * 3^4 * 5^2 * 7$. Therefore, there are $9 * 5 * 3 * 2 = 270$ divisors. Of those divisors, the number that are perfect squares are the ones that only have prime factors that are raised to a power of two; there are $5 * 3 * 2 = 30$. We, however, have overcounted the number those that are a perfect cube. In order to be a perfect cube and a perfect square, you must be a perfect 6th power; there are 2 of these, 1 and 2^6. Therefore, the desired probability is $\frac{30-2}{270} = \frac{28}{270} = \frac{14}{135}$. Therefore, the answer is $\boxed{149}$

24. We extend sides CB and DA through B and A, respectively, to form triangle PDC. Let $\angle ADC$ have measure x. Then $\angle ABC = 2x$ so $\angle ABP$ has angle $180-2x$. Since sides AB and CD are parallel, triangles PAB and PDC must be similar. Thus $\angle PAB = \angle PDC = x$. Using the fact that the angles in a triangle add to $180°$, we can calculate that the measure of $\angle APB = x$. This gives us that $\angle PAB = \angle APB = x$, so $\triangle PAB$ is isosceles, giving us that $PB = AB = 4$. Since $\triangle PDC$ is also isosceles, $CD = PC = PB + BC = 4 + 10 = \boxed{14}$.

25. The 1st and 5th digits are always the same, as well as the 2nd and 4th digits. There are 9 choices for the 1st/5th digit (1, 2, ... , 9), 10 choices for the 2nd/4th digit (0, 1, ... , 9), and 10 choices for the 3rd digit. $9 * 10 * 10 = \boxed{900}$.

26. The area between the squares is $x^2 - y^2 = (x-y)(x+y) = 2020$. Because $x = \frac{(x-y)+(x+y)}{2}$, (x-y) and (x+y) must have the same parity. They cannot both be odd because their product is even, so they are both even. The prime factorization of 2020 is $2 * 2 * 5 * 101$. Knowing the $(x-y) < (x+y)$, we have that the only options are x-y=2 and x+y=1010 or x-y=10 and x+y=202. With the first system of equations, we have that x=506 and the second system gives us that x=106. Of these, the minimum value is $\boxed{x = 106}$.

27. There are only a few ways for three consecutive numbers to form an arithmetic series: 123, 234, 345, or 135. For a permutation to have any one of these, either the arithmetic series must be in increasing or decreasing order (123 or 321). Then, the series can go in any of three spots (**123, *123*, 123**), and there are 2 choices for arranging the last two digits. Thus, there are $2 \cdot 3 \cdot 2 = 12$ permutations with one of these series, for a total of 48. However, we have double-counted some of them: We've counted all permutations with 1234 and 2345 twice. There are 4 permutations with one of these (2 choices for whether it is increasing or decreasing, and 2 choices for where it goes in the permutation). So, there are $48 - 4 - 4 = 40$ permutations with 3 consecutive numbers forming an arithmetic series. Since there are 120 permutations in total, the answer is $120 - 40 = \boxed{80}$.

28. Note that the next letter has to be either a b or an a. If it's a b, then the only option after that would be an a, then a c, then a b, etc. This is clearly a cyclic pattern, hence the case where the fourth letter is b gives one case of a word length 2020. If the fourth letter is a, then the only two options after that are c and b, where c will fall into a similar cycle as b did before. For the b case, there are only two options, a or

c, the a case falling into cycle. And note that now we are back to where we started, well now at $abcabc$, it's not hard to notice that only the most recent occurrences of a, b, and c matter in choosing the next letter. Since in choosing every letter we generate an extra solution, the total number of words with 2020 letters is $\boxed{2018}$.

29. We try to solve the equation based on cases. If x is an integer, then $\lfloor x \rfloor = \lceil x \rceil = x$, and so we get the equation

$$7x + 2x = c,$$

so $x = \frac{c}{9}$. Since x is an integer, this solution is valid if and only if c is a multiple of 9.

If x is not an integer, then $\lceil x \rceil = \lfloor x \rfloor + 1$, so we get the equation

$$7\lfloor x \rfloor + 2(\lfloor x \rfloor + 1) = c,$$

so $\lfloor x \rfloor = \frac{c-2}{9}$. Since $\lfloor x \rfloor$ must be an integer, thus solutions are only valid for x if and only if $c - 2$ is a multiple of 9.

Putting everything together, we see that in the interval $[0, 1000]$, there are 112 multiples of 9 and 111 integers which are 2 more than a multiple of 9, for a total of $112 + 111 = \boxed{223}$ possible values of c.

30. Let such a number be x, and let p be its largest prime factor, such that $x = kp$. Then, we have $kp = 105+p$, so $p(k-1) = 105$. So, p is equal to either 3, 5, or 7. If $p = 3$, then we find that $k = 36$, which is possible. If $p = 5$, then $k = 22$, which is impossible since 22 is divisible by 11, which is larger than 5. If $p = 7$, then $k = 16$, which is also possible. So, the final answer $3 \times 36 + 7 \times 16 = \boxed{220}$.

31. It's enough to just consider the top right hand corner's positioning in a unit cube (by translating the cube such that this is true). Each dimension affects the expected number of points independently. If the x coordinate is less than $\frac{1}{3}$, then there will be 3 possible x-coordinates of lattice points inside the cube, and if the coordinate is bigger than $\frac{1}{3}$ then there will only be 2. This means the expected value of how many x-coordinates fit is $\frac{1}{3} \cdot 3 + \frac{2}{3} \cdot 2 = \frac{7}{3}$. By properties of expected value, to find the amount of 3-D points, we just cube this to get $\frac{343}{27}$, making our answer $\boxed{370}$. Note that we practically derived the result that a cube of volume x in the coordinate space has an expected x lattice points inside of it.

32. Let point E be on the left of the diagram such that the measure of the angle AEB = 60 degrees and AE = EB = AB = 2. Triangle AEC is similar to triangle DBC, and using the ratio AE/BC = 2/3 = BD/BC, we find that m/n = 2/3, meaning that m+n=$\boxed{5}$.

33. First, we calculate how many six-digit sandwich numbers there are. The first three digits of these numbers must be three of $0, 1, 2, ..., 9$. We notice that after we determine the digit triple, the order is already determined. Therefore, there are $\binom{10}{3} = 120$ choices. Similarly, there are also 120 choices for the last three digits. This gives us 120^2 six-digit sandwich numbers. If we want a palindrome, the first three digits must be in the exact opposite order of the last three. Therefore, if we know the first three digits, we have already determined the last three. Therefore, there are 120 six-digit sandwich palindromes. This gives us a fraction of $\frac{120}{120^2} = \frac{1}{120}$ so our final answer is $\boxed{121}$.

34. Let the number of big scoops be b and the number of small scoops be s. We have $30b + 10s = 1000$ or

$$3b + s = 100.$$

We see that since both b and s are whole numbers, $100 - s$ must be a multiple of 3. This gives us solutions $(s, b) = (100, 0), (97, 1), (94, 2), \ldots, (1, 33)$. Now, we have 33 solutions.

However, we know that $s = 100 - 3b$ is not divisible by b. In order for $100 - 3b$ to not divide evenly (or be a multiple of) b, b cannot be a factor of 100. Therefore, $b \neq 1, 2, 4, 5, 10, 20, 25, 50, 100$. However, we see that the maximum value of b is 33. If $b > 33$, we would have $30b > 1000$ which is a contradiction. Therefore, the only values we have to consider are $b \neq 1, 2, 4, 5, 10, 20, 25$. These 7 out of our original 33 do not work. Hence, we have $33 - 7 = \boxed{26}$ ways.

35. If we draw the boxes on the coordinate plane, and imagine the wall as the y-axis and the floor as the x-axis, the plank represents the line connecting the points $(12, 8)$, the corner of the bottom box, and $(4, 14)$, the corner of the top box. The equation for this line is

$$y = -\frac{3}{4}x + 17$$

Since the width of our largest possible box is fixed at 3 inches, we need only consider the area of its base, which can be expressed as xy. Substituting for y, we get

$$xy = -\frac{3}{4}x^2 + 17x$$

We need to maximize this quadratic. All quadratics are maximized at their axis of symmetry: $x = -\frac{b}{2a} = \frac{-17}{-\frac{3}{2}} = \frac{34}{3}$.
When $x = \frac{34}{3}$, $xy = \frac{34}{3} \cdot (-\frac{3}{4} \cdot \frac{34}{3} + 17) = \frac{34}{3} \cdot \frac{17}{2} = \frac{289}{3}$. The volume is $3 \cdot xy$, which maximizes at $3 \cdot \frac{290}{3} = \boxed{289}$.

36. Each male has $m - 1$ male relatives, and each female has m male relatives. So, the average number of male relatives is

$$\frac{m(m-1) + f(m)}{m+f} = \frac{m(m+f-1)}{m+f} = \frac{27}{4}.$$

This implies that $\frac{4m(m+f-1)}{m+f}$ is an integer. However, since $m+f-1$ and $m+f$ are relatively prime, $4m$ must be divisible by $m+f$. If $4m = k(m+f)$ for $k \geq 4$, we find that $f \leq 0$, which is impossible since there is at least one female (the mother). So, either $m = 3f$, $m = f$, or $3m = f$. In the first case, our equation simplifies to $3(m+f-1) = 27$, so $m+f-1 = 9$. This does not have a solution since $m = 3f$. The second case simplifies to $2(m+f-1) = 27$, which does not have a solution either. Finally, the third case simplifies to $m+f-1 = 27$. Thus, $m = 7$, $f = 21$. The answer is $\boxed{217}$.

37. Probabilities that Yul will take some number of hours before she picks up her phone:

$$\frac{1}{3} \text{ hr} \implies \frac{1}{4}.$$

$$\frac{2}{3} \text{ hr} \implies \frac{3}{4} * \frac{1}{2} = \frac{3}{8}.$$

$$1 \text{ hr} \implies \frac{3}{4} * \frac{1}{2} * 1 = \frac{3}{8}.$$

After 1 hour, Yul is guaranteed to have picked up her phone at least once. So the expected amount of hours she will take before she picks up her phone is $\frac{1}{3} * \frac{1}{4} + \frac{2}{3} * \frac{3}{8} + 1 * \frac{3}{8} = \frac{17}{24}$. Since there are 16 hours, $\frac{16}{\frac{17}{24}} = \frac{384}{17}$; $384 + 17 = \boxed{401}$.

38. Let AC and BO intersect at point D. Note that since $AO \parallel BC$, we have that $\triangle ADO \sim \triangle CDB$ with ratio 4 to 5 (the ratio of AO to CB). Thus, $OD : DB = 4 : 5$, and we find that $OD = \frac{4}{4+5} \cdot 4 = \frac{16}{9}$, and $BD = \frac{5}{4+5} \cdot 4 = \frac{20}{9}$.
Now, extend line OB past O to meet the circumcircle at point E. Then, by using the Power of a Point Theorem on point D, we have

$$BD \cdot DE = AD \cdot DC \implies \frac{20}{9} \cdot \left(\frac{16}{9} + 4\right) = AD \cdot \frac{5}{4}AD.$$

Thus, $AD = \sqrt{\frac{4}{5} \cdot \frac{52 \cdot 20}{81}} = \frac{8\sqrt{13}}{9}$, and $DC = \frac{10\sqrt{13}}{9}$. Finally, we have that $AC = \frac{8\sqrt{13} + 10\sqrt{13}}{9} = 2\sqrt{13}$, so the desired answer is $\boxed{52}$.

39. The value of BA_{2b} is $B \cdot 2b + A = 2Bb + A$. Similarly, the value of AB_b is $Ab + B$. So, we have

$$9(Ab + B) = 2Bb + A \implies 9Ab + 9B = 2Bb + A.$$

We can rearrange this to $A(9b - 1) = B(2b - 9)$. Now, if $9b - 1$ and $2b - 9$ are relatively prime, then the smallest possible values for A and B are $2b - 9$ and $9b - 1$ respectively. However, since A and B are less than b, this is not possible. Thus, the GCD of $9b - 1$ and $2b - 9$ is greater than 1. We can compute their GCD using the Euclidean Algorithm:

$$\begin{aligned}
\gcd(9b - 1, 2b - 9) &= \gcd(2b - 9, 9b - 1 - 4(2b - 9)) \\
&= \gcd(2b - 9, b + 35) \\
&= \gcd(b + 35, 2b - 9 - 2(b + 35)) \\
&= \gcd(b + 35, -79).
\end{aligned}$$

Note that 79 is prime. So, if the GCD of $9b-1$ and $2b-9$ is greater than 1, then $b+35$ must be a multiple of 79. So, we must have

$$b \equiv -35 \pmod{79}.$$

So, $b \equiv 44 \pmod{79}$. Thus the smallest possible value of b is $\boxed{44}$. We can plug this back into our equation to check that this works; we get $395A = 79B \implies 5A = B$. So, the solutions for A and B for $b = 44$ are $(A, B) = (1, 5), (2, 10) \ldots, (8, 40)$.

40. First notice that $\angle ECB = \angle ECD = \angle CEB$, since $AB \parallel CD$. So, triangle EBC is isosceles, and $EB = BC$. Similarly, $FA = AD$. Thus, we have that

$$EF = AD + BC - AB = CD,$$

since $EFCD$ is a parallelogram. Hence, we have that $AD + BC = AB + CD$. By the converse of Pitot's Theorem, this implies that $ABCD$ is a tangential quadrilateral.

It follows that point P is the incenter of the trapezoid, and lies on the angle bisectors of $\angle DAB$ and $\angle ABC$ as well. Now, note that since $\angle BAD + \angle ADC = 180$, we have that $\angle PAD + \angle PDA = 90 \implies \angle APD = 90$. This means that APD is a right triangle and so by the Pythagorean Theorem, $PD = 6$. Now, the inradius of the trapezoid is equal to the length of the altitude from point P in triangle APD, which is equal to $\frac{6 \cdot 8}{10} = \frac{24}{5}$.

Finally, the area of the trapezoid is equal to the inradius times the semiperimeter, so

$$\frac{24}{5} \cdot \frac{1}{2}(AB + BC + CD + DA) = \frac{24}{5} \cdot (AD + BC) = 216.$$

It follows that $AD + BC = 45$, and $BC = \boxed{35}$.