Joe Holbrook Memorial Math Competition

8th Grade Solutions

October 18, 2020

1. Note that one of the 5^2 terms in $\frac{5^2 \times 5^2}{5^2}$ can be reduced, meaning that the first part of our expression can be simplified to $5^2 = 25$. For the second part, we can break the fraction $\frac{5^2 + 5^2}{5^2}$ into $\frac{5^2}{5^2} + \frac{5^2}{5^2} = 1 + 1 = 2$. Therefore, our final sum is $25 + 2 = \boxed{27}$.

2. Buying 7 pencils costs $7 \cdot 85 = 595$ cents. You buy 3 more pencils than erasers, so you buy $7 - 3 = 4$ erasers. Buying 4 erasers costs $4 \cdot 92 = 368$ cents. Thus in total you spend $595 + 368 = \boxed{963}$ cents.

3. The number of minutes from 4:37pm to 5:52 pm is $15 + 60 = 75$. Adding the additional 87 minutes, Frank will have $75 + 87 = \boxed{162}$ total minutes to trick-or treat.

4. First, convert Suzys into Snowmans to get $24 \cdot \frac{5}{2} = 60$ Snowmans. Now convert Snowmans into Dougs to get $60 \cdot \frac{14}{3} = \boxed{280}$ Dougs.

5. For 2 standard 6-sided dice, the probability of rolling a sum of 7 is $\frac{6}{36}$. However, with the changed dice, two new options now give 7: $3, 4$ and $4, 3$. So now, the probability is $\frac{8}{36} = \frac{2}{9}$ and the answer is $\boxed{11}$.

6. We know that one dog has $(2x + 5) \cdot 3 = 6x + 15$ toes and one cat has $(3x - 3) \cdot 5 = 15x - 15$ toes. Then 3 dogs have a total of $3(6x + 15) = 18x + 45$ toes, and 2 cats have a total of $2(15x - 15) = 30x - 30$ toes. We know that there are a total of 111 toes, so

$$18x + 45 + 30x - 30 = 111.$$

Simplifying this equation gives $48x = 96$, and dividing this by 48 gives $x = \boxed{2}$.

7. 61 and 67 are prime numbers so they only have 2 factors. 62 $(2 \cdot 31)$ and 65 $(5 \cdot 13)$ each have 4 factors. 63, 64, and 66 does not satisfy Kevin's conditions as they have 6, 7, and 8 factors respectively. Thus there are $\boxed{4}$ days where Kevin stayed home.

8. We want to isolate x, which we can do by raising both sides of the equation to the $\frac{3}{5}$ power, as this will give us $(x^{\frac{5}{3}})^{\frac{3}{5}} = x = 32^{\frac{3}{5}}$. We now have $x = 32^{\frac{3}{5}}$, which means that $x = \sqrt[5]{32^3} = \sqrt[5]{(2^5)^3} = 2^3 = \boxed{8}$.

9. Growing the area by a factor of 72 means increasing both the height and width by $\sqrt{72} = 6\sqrt{2}$. Then the width is $10 \cdot 6\sqrt{2} = 60\sqrt{2}$ while the length is $20 \cdot 6\sqrt{2} = 120\sqrt{2}$. Therefore, $a+b+c+d = 60+2+120+2 = \boxed{184}$.

10. We see that if $a^2 < n \leq (a+1)^2$ for some integer a, then $a < \sqrt{x} \leq a + 1$, so a is the least integer greater than or equal to x. Proceeding from this:

For $5 \leq n \leq 9$, $\lceil \sqrt{n} \rceil = 3$. There are 5 values of 3 in this range.

For $10 \leq n \leq 16$, $\lceil \sqrt{n} \rceil = 4$. There are 7 values of 4 in this range.

For $17 \leq n \leq 25$, $\lceil \sqrt{n} \rceil = 5$. There are 9 values of 5 in this range.

For $26 \leq n \leq 29$, $\lceil \sqrt{n} \rceil = 6$. There are 4 values of 6 in this range.

Our total sum is $5 \cdot 3 + 7 \cdot 4 + 9 \cdot 5 + 4 \cdot 6 = \boxed{112}$

11. Isaac's first hint tells us that his parents' ages have either 3 or 4 factors. If you play around with the prime factorizations of numbers, you will find that the only numbers with 3 factors are the squares of primes, and the only numbers with 4 factors are the products of 2 distinct primes.
Isaac's second hint restricts his parents' ages to between 20 and 50. Thus, Isaac's parents can only be 21, 22, 25, 26, 33, 34, 35, 38, 39, 46, 49. Isaac also tells us his parents' ages end in 9, so they can only be 39 or 49. Since Isaac's dad is older than his mom, we know his dad is 49 and his mom is 39. $39 + 49 = \boxed{88}$.

12. We see that in order to be divisible by both 6 and 4, the number has to be divisible by their least common multiple. Therefore, we're looking for numbers that are divisible by 12 but not $12 \cdot 7 = 84$. There are $\lfloor \frac{1000}{12} \rfloor = 83$ numbers less than 1000 that are divisible by 12. Of those 83, $\lfloor \frac{83}{7} \rfloor = 11$ are divisible by 7. Therefore, there are $84 - 11 = \boxed{73}$ numbers.

13. Let the common external tangent be tangent to circle A at D and circle B at C. AD and BC are perpendicular to CD due to the tangency condition. Quadrilateral ABCD can be split into a rectangle and a right triangle. The legs of the right triangle are $6-4 = 2$ and 12, so $AB = \sqrt{2^2 + 12^2} = 2\sqrt{37} \implies 2+37 = \boxed{39}$.

14. Notice that because the radius EO is perpendicular to CD, $DE = CE \implies \angle CDE = \angle DCE = \left(\frac{180 - 100}{2}\right)^\circ = 40^\circ \implies \angle ADC = (90 - 40)^\circ = 50^\circ$. Since opposite angles of a cyclic quadrilateral are supplementary, $\angle ABC = (180 - 50)^\circ = \boxed{130^\circ}$.

15. WLOG, assume that $AD = 3$ and $DC = 7$. By the angle bisector theorem, $\frac{BC}{AB} = \frac{7}{3}$. If $BC = 7$ and $AB = 3$, the triangle is degenerate, and if $BC = 14$ and $AB = 6$, the triangle inequality holds. Hence, the answer is $14 + 10 + 6 = \boxed{30}$.

16. Since a total of 4 edges will be part of the top face, and each color is on the same number of edges, we just divide 4 by the number of colors, 3, to get $\frac{4}{3}$, giving $a + b = \boxed{7}$.

17. Note that $126 = 2 \cdot 3^2 \cdot 7$. The question is equivalent to finding the values of n for which $\frac{n}{2}$ divides 126, but n doesn't. So, the exponent of 2 in the prime factorization of n must be 2. The other prime factors of n must be 3 and 7, and their exponents must be less than 2 and 1, respectively. So, the possibilities for n are $4 \cdot 1$, $4 \cdot 3$, $4 \cdot 7$, $4 \cdot 9$, $4 \cdot 21$, and $4 \cdot 63$. However, note that the last one doesn't work because n must be less than 126. The other 5 work. Their sum is $4(1 + 3 + 7 + 9 + 21) = \boxed{164}$.

18. The area between the squares is $x^2 - y^2 = (x - y)(x + y) = 2020$. Because $x = \frac{(x-y)+(x+y)}{2}$, $x - y$ and $x + y$ must have the same parity. They cannot both be odd because their product is even, so they are both even. The prime factorization of 2020 is $2 \cdot 2 \cdot 5 \cdot 101$. Knowing that $x - y < x + y$, we have that the only options are $x - y = 2$ and $x + y = 1010$ or $x - y = 10$ and $x + y = 202$. With the first system of equations, we have that $x = 506$ and the second system gives us that $x = 106$. Of these, the minimum value is $x = \boxed{106}$.

19. There are only a few ways for three consecutive numbers to form an arithmetic series: 123, 234, 345, or 135. For a permutation to have any one of these, either the arithmetic series must be in increasing or decreasing order (123 or 321). Then, the series can go in any of three spots (**123, *123*, 123**), and there are 2 choices for arranging the last two digits. Thus, there are $2 \cdot 3 \cdot 2 = 12$ permutations with one of these series, for a total of 48. However, we have double-counted some of them: We've counted all permutations with 1234 and 2345 twice. There are 4 permutations with one of these (2 choices for whether it is increasing or decreasing, and 2 choices for where it goes in the permutation). So, there are $48 - 4 - 4 = 40$ permutations with 3 consecutive numbers forming an arithmetic series. Since there are 120 permutations in total, the answer is $120 - 40 = \boxed{80}$.

20. We can apply Simon's Favorite Factoring Trick to each of the equations.

$$xy + 2x + 3y = 6,$$
$$yz + 4y + 2z = 6,$$
$$xz + 4x + 3z = 30,$$

Adding 6, 8, and 12 to both sides of each equation

$$xy + 2x + 3y + 6 = (x + 3)(y + 2) = 12,$$
$$yz + 4y + 2z + 8 = (y + 2)(z + 4) = 14,$$
$$xz + 4x + 3z + 12 = (x + 3)(z + 4) = 42,$$

If we take the product of all three equations, we obtain

$$[(x+3)(y+2)(z+4)]^2 = 12 \cdot 14 \cdot 42 = 2^4 \cdot 3^2 \cdot 7^2,$$

so

$$(x+3)(y+2)(z+4) = \pm 2^2 \cdot 3 \cdot 7.$$

We can now substitute that $(y+2)(z+4) = 14$ to find that

$$(x+3)(y+2)(z+4) = 14(x+3) = \pm 2^2 \cdot 3 \cdot 7.$$

Hence, $x+3 = \pm 6$, so x is 3 or -9. The positive root is $x = \boxed{3}$.

21. We first multiply both sides of the congruence $12^{-1} \equiv 12^2 \pmod{m}$ by 12.

$$\underbrace{12 \cdot 12^{-1}}_{1} \equiv \underbrace{12 \cdot 12^2}_{12^3} \pmod{m}.$$

So we now have that $12^3 - 1 = 1727$ is a multiple of m. We find that the only three-digit factor of 1727 is $\boxed{157}$.

22. Note that the next letter has to be either a b or an a. If it's a b, then the only option after that would be an a, then a c, then a b, etc. This is clearly a cyclic pattern, hence the case where the fourth letter is b gives one case of a word length 2020. If the fourth letter is a, then the only two options after that are c and b, where c will fall into a similar cycle as b did before. For the b case, there are only two options, a or c, the a case falling into cycle. And note that now we are back to where we started, well now at *abcabc*, it's not hard to notice that only the most recent occurrences of a, b, and c matter in choosing the next letter. Since in choosing every letter we generate an extra solution, the total number of words with 2020 letters is $\boxed{2018}$.

23. Because the coordinates of the points are so difficult to work with, it is a very good idea to try and shift the points around to get a triangle whose area is equal to that of ABC, and easier to find.
Recall that shifting point A along a line parallel to BC does not change the area of the triangle. This is because the height of point A with respect to line BC does not change. The slope of line BC is

$$\frac{\sqrt{14} - \frac{\sqrt{7}}{2} - \sqrt{14}}{2\sqrt{7} - \sqrt{7}} = -\frac{1}{2}.$$

Notice that the line connecting A with the origin, $(0,0)$, also has slope $-\frac{1}{2}$. Call the origin A'; the area of ABC is equal to the area of $A'BC$.
Let's repeat this trick with point B. The slope of $A'C$ is $\frac{\sqrt{14}}{\sqrt{7}} = \sqrt{2}$. Using point-slope form, the line parallel to $A'C$ running through B is

$$y - \left(\sqrt{14} - \frac{\sqrt{7}}{2}\right) = \sqrt{2}(x - 2\sqrt{7})$$

Let's choose point B' on the this line and the x-axis. Plugging in $y = 0$, we get that $x = \sqrt{7} + \frac{\sqrt{14}}{4}$. Note that the area of $A'B'C =$ the area of $A'BC$, which equals the area of ABC. Now we can use $A = \frac{1}{2}Bh$ to find the area of $A'B'C$.

$$A = \frac{1}{2}(\sqrt{7} + \frac{\sqrt{14}}{4})(\sqrt{14}) = \frac{7 + 14\sqrt{2}}{4}$$

$a = 7, b = 14, c = 2, d = 4 \Rightarrow a + b + c + d = \boxed{27}$.

24. We do a little bit of casework. If indeed some permutation of $\triangle ACD$ is similar to $\triangle ABD$, then $\angle ADC$ is equal to either $\angle ABD$, $\angle BAD$, or $\angle BDA$. If the first option is true, then AB and AD are parallel by the corresponding angles theorem, something which is just false. For the second option, AB and BC are parallel by the alternate interior angles theorem, which is also just false. For the third option, since the sum of $\angle ADC$ and $\angle BDA$ is 180, then both will have to be 90, which is a working case due to altitude similarity conditions. Hence the answer is just $\boxed{90}$.

25. If we draw a diagram showing the positions of Anthony (A), David (D), and Erez (E), and all the space within 6 feet of any one of them, we get three overlapping circles with radius 6, each running through the centers of the other two. We wish to find the area of the figure.
One clever way to find the total area is to use the principle of Inclusion-Exclusion, which allows us to rewrite the total area as the sum of a few component parts. Notice that if we simply add the areas of circles A, D, and E, we double-count the areas of regions where exactly 2 circles overlap, and triple-count the area where 3 circles overlap. We need to subtract out these areas accordingly.
Notice that the central region, the area shared by three circles, is made up of an equilateral triangle with side length 6, and 3 segments of a circle. In particular, notice that each segment's area can be expressed as $\frac{1}{6}$ the area of the circle minus the area of the equilateral triangle. It follows that

$$\begin{aligned} A_{\text{central area}} &= \frac{1}{2}\cdot 6\cdot 3\sqrt{3} + 3(\frac{1}{6}6^2\pi - \frac{1}{2}\cdot 6\cdot 3\sqrt{3}) \\ &= 9\sqrt{3} + 18\pi - 27\sqrt{3} \\ &= 18\pi - 18\sqrt{3} \end{aligned}$$

The remaining three sections, where exactly 2 circles overlap, have equal area. Notice that these sections can be formed by attaching 2 circle segments onto an equilateral triangle with side length 6, and carving a third out the bottom. Thus, this area can be expressed as

$$\begin{aligned} A_{\text{ section}} &= \frac{1}{2}\cdot 6\cdot 3\sqrt{3} + 2(\frac{1}{6}\cdot 6^2\pi - \frac{1}{2}\cdot 6\cdot 3\sqrt{3}) - (\frac{1}{6}\cdot 6^2\pi - \frac{1}{2}\cdot 6\cdot 3\sqrt{3}) \\ &= 9\sqrt{3} + 12\pi - 18\sqrt{3} - 6\pi + 9\sqrt{3} \\ &= 6\pi \end{aligned}$$

Therefore, the total area is

$$\begin{aligned} A_{\text{ total}} &= 3\cdot 6^2\pi - 3\cdot 6\pi - 2(18\pi - 18\sqrt{3}) \\ &= 108\pi - 18\pi - 36\pi + 36\sqrt{3} \\ &= 54\pi + 36\sqrt{3} \end{aligned}$$

Thus, $a = 54, b = 36, c = 3 \Rightarrow a + b + c = \boxed{93}$.

26. We try to solve the equation based on cases. If x is an integer, then $\lfloor x\rfloor = \lceil x\rceil = x$, and so we get the equation $7x + 2x = c$, so $x = c/9$. Since x is an integer, this solution is valid if and only if c is a multiple of 9. If x is not an integer, then $\lceil x\rceil = \lfloor x\rfloor + 1$, so we get the equation $7\lfloor x\rfloor + 2(\lfloor x\rfloor + 1) = c$, so $\lfloor x\rfloor = (c-2)/9$. Since $\lfloor x\rfloor$ must be an integer, thus solutions are only valid for x if and only if $c - 2$ is a multiple of 9.
Putting everything together, we see that in the interval $[0, 1000]$, there are 112 multiples of 9 and 111 integers which are 2 more than a multiple of 9, for a total of $112 + 111 = \boxed{223}$ possible values of c.

27. Let such a number be x, and let p be its largest prime factor, such that $x = kp$. Then, we have $kp = 105+p$, so $p(k-1) = 105$. So, p is equal to either 3, 5, or 7. If $p = 3$, then we find that $k = 36$, which is possible. If $p = 5$, then $k = 22$, which is impossible since 22 is divisible by 11, which is larger than 5. If $p = 7$, then $k = 16$, which is also possible. So, the final answer $3 \times 36 + 7 \times 16 = \boxed{220}$.

28. It's enough to just consider the top right hand corner's positioning in a unit cube (by translating the cube such that this is true). Each dimension affects the expected number of points independently. If the x coordinate is less than $1/3$, then there will be 3 possible x-coordinates of lattice points inside the cube, and if the coordinate is bigger than $1/3$ then there will only be 2. This means the expected value of how many x-coordinates fit is $(1/3)\cdot 3 + (2/3)\cdot 2 = 7/3$. By properties of expected value, to find the amount of 3-D points, we just cube this to get 34327, making our answer $\boxed{370}$. Note that we practically derived the result that a cube of volume x in the coordinate space has an expected x lattice points inside of it.

29. Extend segment CB through B to point E such that $AE = EB = AB = 2$. Triangle AEC is similar to triangle DBC, and using the ratio $AE/EC = 2/3 = BD/BC$, we find that $m/n = 2/3$, meaning that $m + n = \boxed{5}$.

30. First, we calculate how many six-digit sandwich numbers there are. The first three digits of these numbers must be three of $0, 1, 2, ..., 9$. We notice that after we determine the digit triple, the order is already

determined. Therefore, there are $\binom{10}{3} = 120$ choices. Similarly, there are also 120 choices for the last three digits. This gives us 120^2 six-digit sandwich numbers. If we want a palindrome, the first three digits must be in the exact opposite order of the last three. Therefore, if we know the first three digits, we have already determined the last three. Therefore, there are 120 six-digit sandwich palindromes. This gives us a fraction of $\frac{120}{120^2} = \frac{1}{120}$ so our final answer is $\boxed{121}$.

31. Let $f(x) = ax^2 + bx + c$. Then we must have

$$f(x) = ax^2 + bx + c = x^2 + kx + k \cdot (a(x-1)^2 + b(x-1) + c).$$

The coefficients of x^2, x, and the constant must be the same on both sides. Since the only x^2 term in the expansion of $f(x-1)$ is ax^2, we must have $ax^2 = x^2 + kax^2 \implies a = 1 + ka$. So, $k = (a-1)/a$. Now, note that k is a positive integer and a is an integer. The only way for $(a-1)/a$ to be a positive integer for an integer a is $a = -1$, $k = 2$.

Now, our equation is $-x^2 + bx + c = x^2 + 2x + 2(-(x-1)^2 + b(x-1) + c)$. Expanding, we get that the right hand side is equal to $-x^2 + x(2 + 4 + 2b) + (2c - 2b - 2)$. Comparing coefficients on both sides, we have $b = 2 + 4 + 2b$, and $c = 2c - 2b - 2$. The first equation gives us $b = -6$. Plugging this into the second equation, we get $c = -10$. So, we have that $f(x) = -x^2 - 6x - 10$. Finally $|f(1)| = 1 + 6 + 10 = \boxed{17}$.

32. If we draw the boxes on the coordinate plane, and imagine the wall as the y-axis and the floor as the x-axis, the plank represents the line connecting the points $(12, 8)$, the corner of the bottom box, and $(4, 14)$, the corner of the top box. The equation for this line is $y = -\frac{3}{4}x + 17$. Since the width of our largest possible box is fixed at 3 inches, we need only consider the area of its base, which can be expressed as xy. Substituting for y, we get $xy = -\frac{3}{4}x^2 + 17x$. We need to maximize this quadratic. This quadratic is maximized at its axis of symmetry: $x = -\frac{b}{2a} = \frac{-17}{-\frac{3}{2}} = \frac{34}{3}$.
When $x = \frac{34}{3}$, $xy = \frac{34}{3} \cdot \left(-\frac{3}{4} \cdot \frac{34}{3} + 17\right) = \frac{34}{3} \cdot \frac{17}{2} = \frac{289}{3}$. The volume is $3 \cdot xy$, which maximizes at $3 \cdot \frac{289}{3} = \boxed{289}$.

33. Each male has $m - 1$ male relatives, and each female has m male relatives. So, the average number of male relatives is

$$\frac{m(m-1) + f(m)}{m+f} = \frac{m(m+f-1)}{m+f} = \frac{27}{4}.$$

This implies that $\frac{4m(m+f-1)}{m+f}$ is an integer. However, since $m + f - 1$ and $m + f$ are relatively prime, $4m$ must be divisible by $m + f$. If $4m = k(m + f)$ for $k \geq 4$, we find that $f \leq 0$, which is impossible since there is at least one female (the mother). So, either $m = 3f$, $m = f$, or $3m = f$. In the first case, our equation simplifies to $3(m + f - 1) = 27$, so $m + f - 1 = 9$. This does not have a solution since $m = 3f$. The second case simplifies to $2(m + f - 1) = 27$, which does not have a solution either. Finally, the third case simplifies to $m + f - 1 = 27$. Thus, $m = 7$, $f = 21$. The answer is $\boxed{217}$.

34. We calculate the expected value for how long (in minutes) it takes for Yul to pick the phone up once. It is $20 \cdot \frac{1}{4} + 40 \cdot \frac{3}{4} \cdot \frac{1}{2} + 60 \cdot \frac{3}{4} \cdot \frac{1}{2} = \frac{85}{2}$. So, the expected number of times she picks up in 16 hours (960 minutes) is $960/\frac{85}{2} = 384/17$. This gives the final answer of $\boxed{401}$.

35. Let AC and BO intersect at point D. Note that since $AO \parallel BC$, we have that $\triangle ADO \sim \triangle CDB$ with ratio 4 to 5 (the ratio of AO to CB). Thus, $OD : DB = 4 : 5$, and we find that $OD = \frac{4}{4+5} \cdot 4 = \frac{16}{9}$, and $BD = \frac{5}{4+5} \cdot 4 = \frac{20}{9}$.
Now, extend line OB past O to meet the circumcircle at point E. Then, by using the Power of a Point Theorem on point D, we have

$$BD \cdot DE = AD \cdot DC \implies \frac{20}{9} \cdot \left(\frac{16}{9} + 4\right) = AD \cdot \frac{5}{4}AD.$$

Thus, $AD = \sqrt{\frac{4}{5} \cdot \frac{52 \cdot 20}{81}} = \frac{8\sqrt{13}}{9}$, and $DC = \frac{10\sqrt{13}}{9}$. Finally, we have that $AC = \frac{8\sqrt{13}+10\sqrt{13}}{9} = 2\sqrt{13}$, so the desired answer is $\boxed{52}$.

36. We can complete the square in A as follows:

$$A = \sqrt{-x^2+4x+32} = \sqrt{-(x^2-4x-32)} = \sqrt{-((x-2)^2-36)} = \sqrt{-(x-2)^2+36}.$$

Similarly, we do the same for B:

$$B = \sqrt{-x^2-4x} = \sqrt{-(x^2+4x)} = \sqrt{-((x+2)^2-4)} = \sqrt{-(x+2)^2+4}.$$

Let us plot the graph of A on the Cartesian plane by letting $y = \sqrt{-(x-2)^2+36}$. Squaring gives $y^2 = -(x-2)^2+36 \implies (x-2)^2+y^2 = 36$. That is the equation of a circle with center $(2,0)$ and radius 6. However, note that we squared the equation in the beginning, so we check for extraneous solutions. We note that y must be positive, so this graph is actually only the top half of the circle above the x-axis.

We can do the same for B. Letting $y = \sqrt{-(x+2)^2+4}$, we get that $y^2 = -(x+2)^2+4 \implies (x+2)^2+y^2 = 4$, which is an equation of a circle with center $(-2,0)$ and radius 2. However, again note that $y \geq 0$, so the graph of B is just the top half of this circle.

We can interpret $A-B$ as the vertical distance between these semicircles at a fixed point x(the distance between the y coordinates). Clearly, $x \in [-4,0]$ because of the domain for which both graphs are defined, and it is easy to note that the maximum value of $A-B$ occurs at $x=0$. Therefore, we can plug $x=0$ into both equations to find that $A-B = \sqrt{32} \implies \boxed{n=32}$.

37. $f(x)$ is an increasing function for positive x, so there is at most 1 solution. Assume t is the solution (if it exists). Then let $m = 31/t$. Now $m = \lfloor t \cdot \lfloor t \cdot \lfloor t \rfloor \rfloor \rfloor$, so m is an integer. Additionally, since f is an increasing function and $f(2) = 16 < 31 < 81 = f(3)$, we know that $2 < t < 3$. So, $2 < 31/m < 3$, which implies that $m = 11, 12, 13, 14, \text{or} 15$. Checking each of these, we find that $x = 31/12$ is a solution, so the answer is $31+12 = \boxed{43}$.

38. The value of BA_{2b} is $B \cdot 2b + A = 2Bb + A$. Similarly, the value of AB_b is $Ab+B$. So, we have

$$9(Ab+B) = 2Bb+A \implies 9Ab+9B = 2Bb+A.$$

We can rearrange this to $A(9b-1) = B(2b-9)$. Now, if $9b-1$ and $2b-9$ are relatively prime, then the smallest possible values for A and B are $2b-9$ and $9b-1$ respectively. However, since A and B are less than b, this is not possible. Thus, the GCD of $9b-1$ and $2b-9$ is greater than 1. We can compute their GCD using the Euclidean Algorithm:

$$\begin{aligned} \gcd(9b-1, 2b-9) &= \gcd(2b-9, 9b-1-4(2b-9)) = \gcd(2b-9, b+35) \\ &= \gcd(b+35, 2b-9-2(b+35)) = \gcd(b+35, -79). \end{aligned}$$

Note that 79 is prime. So, if the GCD of $9b-1$ and $2b-9$ is greater than 1, then $b+35$ must be a multiple of 79. So, we must have

$$b \equiv -35 \pmod{79}.$$

So, $b \equiv 44 \pmod{79}$. Thus the smallest possible value of b is $\boxed{44}$. We can plug this back into our equation to check that this works; we get $395A = 79B \implies 5A = B$. So, the solutions for A and B for $b = 44$ are $(A,B) = (1,5), (2,10)\dots, (8,40)$.

39. Adding n^2 to both sides of $n^2 a_{n+1} = (n+1)^2 a_n + 2n + 1$ gives $n^2(a_{n+1}+1) = (n+1)^2 a_n + (n+1)^2 = (n+1)^2(a_n+1)$. Since $a_1 + 1 = 1$ and $a_{n+1} + 1 = \frac{(n+1)^2}{n^2} \cdot (a_n+1)$, we can see that $a_n = n^2 - 1$. So, the answer is $a_{50} = 50^2 - 1 = \boxed{2499}$.

40. We claim that only squarefree numbers work.
If n is squarefree, then it can be written as $n = p_1 p_2 .... p_t$. Take $k = (p_1-1)(p_2-1)....(p_t-1)+1$. This works, by Fermat's Little Theorem.

Assume p^2 divides n. Plug in $y=1$ to get $x^k + 1 = (x+1)^k$. Plug in $x=1$ to get $2 = 2^k (mod n)$. By induction, we find that $3^k \equiv 3 \pmod{n}, 4^k \equiv 4 \pmod{n}, \dots x^k \equiv x \pmod{n}$ by induction. So $p^k \equiv p \pmod{n}$, which implies $p^{k-1} \equiv 1 \pmod{n/p}$. But if $k > 1$ and n/p is an integer that is divisible by p, the last equation is impossible. So contradiction, we cannot have any square factors in n.

We can find that there are 38 numbers between 2 and 99 that are divisible by a square number, so there are $98 - 38 = \boxed{60}$ remaining squarefree numbers.