Joe Holbrook Memorial Invitational Competition (JHMIC)

8th Grade

March 28, 2021

General Rules

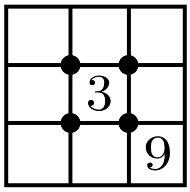
- You will have **90 minutes** to solve **16 questions**. Your score is the **sum** of the **point values** of the questions for which you got a correct answer. There are a total of 100 points.
  - Questions **1-5** are each worth **5 points**
  - Questions **6-10** are each worth **6 points**
  - Questions **11-13** are each worth **7 points**
  - Questions **14-16** are each worth **8 points**
- Only answers recorded on the appropriate Google Form will be graded.
- You are to remain visible to your proctor at **all times**. Please have your video camera on during the exam.
- This is an individual test. Anyone caught communicating with another student or using technology in an inappropriate way will be removed from the exam.
- You may not use the following aids:
  - Calculator or other computing device
  - Compass
  - Protractor
  - Ruler or straightedge

Other Notes

- All answers are positive integers. Please enter them with no spaces in between into the Google Form.
- Do not include commas in your answers. For example, the number one thousand is to be entered 1000 not $1,000$.
- You must not write units in your answers.
- Ties will be broken by the number of correct responses. Further ties will be broken by last question answered correctly, then second-to-last, and so on.

1. *[5 pts]* The probability that when a fair six sided die is rolled twice and the numbers rolled are summed, the result is prime, can be written as $\frac{a}{b}$ in simplest terms. What is $a+b$?

2. *[5 pts]* Fill in the other seven numbers in the following 3 by 3 grid with the numbers from 1-9 excluding 3 and 9 such that the sum of the numbers in each row is equal and the sum of the numbers in each column is equal and each number is used exactly once. Let a be the product of the four numbers in the corners and b be the sum of the four numbers in the corners. What is $a+b$?



3. *[5 pts]* Let ABC be an isosceles triangle with $AB = AC$. Let D be the midpoint of BC, and let E be the midpoint of DC. If $AB = 7$ and $AE = 4$, then compute the square of the area of ABC.

4. *[5 pts]* Let $\S(k)$ be defined as the number of integers $1 \le n \le k$ such that $\gcd(k, n) = 1$. Evaluate:

$$\S(1) + \S(43) + \S(47) + \S(2021) + 2021$$

5. *[5 pts]* The five distinct numbers a, b, c, d, e satisfy the property that the mean of the medians of the five sets $\{a, b, c, d\}, \{a, b, c, e\}, \{a, b, d, e\}, \{a, c, d, e\}$, and $\{b, c, d, e\}$ is equal to 1.2 times the median of the five means of these same sets. Given that the median of the original set is 51, what is the sum of its maximum and minimum element?

6. *[6 pts]* Greg writes the numbers 1, 2, 3, 4, 5 onto a blackboard. He writes the expression $(ab + c)d + e$ onto the board. He tells Nikhil to randomly substitute in the numbers into the five variables such that every number is used. The probability Nikhil's number is even can be written in the form $\frac{x}{y}$ in simplest terms. What is $x + y$?

7. *[6 pts]* Let there be a clock with two hands, one with length 2 and one with length 1. Greg picks a random time to look at the clock. The probability the area of the triangle formed by the two clock hands at that time is at least $\frac{1}{2}$ can be written as $\frac{a}{b}$, where a, b have a greatest common divisor of 1. What is $100a + b$?

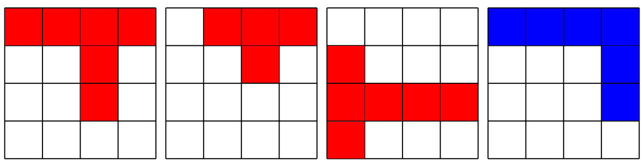
8. *[6 pts]* Let there be a 4 by 4 grid. Define a T-shape as the union of a 1 by m strip and a n by 1 strip such that

   (a) the two strips are perpendicular

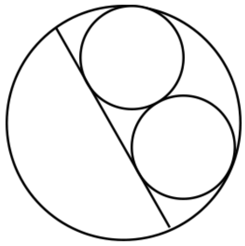
   (b) the two strips share one cell that is the endpoint of one of the strips but not the other

   (c) both strips have a length of at least 2

   How many T-shapes are there in the grid? The red shapes below illustrate some valid T-shapes, while the blue shape is not a valid T-shape.



9. *[6 pts]* A circle of radius 50 has two smaller circles of equal radii that are internally tangent to it and externally tangent to each other. The bigger circle has a chord that is externally tangent to both smaller circles and π times its length is the sum of the perimeters of the smaller circles. What is the radius of one of the smaller circles?



10. *[6 pts]* How many triples of positive integers (x, y, z) satisfy $xyz + yz + z = 36$?

11. *[7 pts]* There exists a unique prime p such that there exists a positive integer x for which p divides both $x^3 - 8$ and $x^2 - 5$. Compute p.

12. *[7 pts]* What is the expected value of the area of a randomly selected square from all the squares that have a diagonal with negative slope and endpoints $(x_1, y_1), (x_2, y_2)$ such that $0 \leq x_1, x_2, y_1, y_2 \leq 8$ and x_1, x_2, y_1, y_2 are integers?

13. *[7 pts]* Let $ABCD$ be a trapezoid with $AB \parallel CD$. There exists a point P such that circle ω centered at P is tangent to all four sides of $ABCD$. Specifically, ω is tangent to AB and AD at X and Y respectively, such that $XY = 6$ and $PD = 30$. What is the smallest possible area of $ABCD$?

14. *[8 pts]* Let $f(x) = ax^2 + bx + c$. If $f(-2) = f(4) = f(a) = f(b) = f(c)$, what is the absolute value of the sum of all possible values of $f(5)$?

15. *[8 pts]* Let $d(n)$ be the number of positive divisors of n. Let $F(n)$ be the sum of the $d(m)$ over all positive divisors m of n. What is the smallest n such that at least four distinct primes divide $F(n)$?

16. *[8 pts]* Let the real numbers x, y, z satisfy $x + y + z > 0$ and the following system of equations:

$$x^2 - yz = 3$$
$$y^2 - xz = 4$$
$$z^2 - xy = 5$$

If $|xyz| = \frac{a}{b}$ in simplest terms, what is $a + b$?