Joe Holbrook Memorial Math Competition

5th Grade Solutions

October 17, 2021

1. By the PEMDAS order of operations: $(7+7)\times((7\div7)\div7)+(7-7)^7 = 14\times(1\div7)+0^7 = 14\times\frac{1}{7}+0 = \boxed{2}$.

2. Let x be the number of windows on her Mac. Then she has $3x$ windows on her PC. In total, she has $x+3x=4x$ computers, so Bianca has $4x=20 \implies x=5 \implies 3x=\boxed{15}$ windows on her PC.

3. The sum simplifies to $5^2+2^5=25+32=\boxed{57}$.

4. In each subsequent day, Catherine drives four more miles. The value in each row increases by 4:

   Day 1: 12 miles
   Day 2: 16 miles
   Day 3: 20 miles
   Day 4: 24 miles
   Day 5: 28 miles

   Therefore, on the 5th day, Catherine will drive $\boxed{28}$ miles.

5. After the 34 dalmatians go missing, there are $101-34=67$ dalmatians left. These can be split into 5 groups of 13 dalmatians with $\boxed{2}$ leftover.

6. We need to calculate $\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{4}{5}\cdot100=\frac{1}{5}\cdot100=\boxed{20}$ cents by cancelling.

7. We can simply test the first few primes. We can see that 2, 3, and 5 don't work, whereas $\boxed{7}$ does (6 and 8 both have 4 divisors).

8. Since there are 20 students, it will take $20\cdot5=100$ seconds to complete the actual pencil sharpening. Additionally, there are 19 exchanges between students that last 2 seconds each, for a total exchange time of $19\cdot2=38$ seconds. Thus, the total time it would take is $100+38=\boxed{138}$ seconds.

9. Pineapple's iPine charges 15% per hour, which means it charges 1% every 4 minutes. Therefore, it will charge 100% in 400 minutes. Jamsung's Jalaxy charges 1% every 6 minutes, so it will charge to 100% in 600 minutes. Thus, the difference in minutes will be $600-400=\boxed{200}$ minutes.

10. We can add probabilities to find the probability that Yul arrives by a certain time. There is a 20% chance she arrives by 5pm, a $20+20=40\%$ chance she arrives by 5:10pm, and $40+30=70\%$ chance she arrives by 5:20pm. Thus, Alicia should arrive at 5:20pm, which is $\boxed{20}$ minutes late.

11. In essence, we are trying to find the diameter of the circular park, as they are walking the path created from the opposite ends of the circle. Because both friends walked 18π miles, they walked 36π miles in total, implying that the circumference of the circle is 36π. The circumference of a circle is equal to its diameter times π, so the diameter is is $\boxed{36}$.

12. Janice is currently 12 years old, and John is currently 24 years old. When John is 50 years old, he will be 26 years older than his current age. This means that Janice will also be 26 years older than her current age, so she will be $12+26=\boxed{38}$ years old.

13. If the answer is a one-digit number, then the answer would be $1+11=12$, which is not a one-digit number. If the answer is a two-digit number, the answer would be $2+11=13$, which is indeed a two-digit number. Thus, the answer is $\boxed{13}$.

14. Since the area of one lilypad is π square feet, the total area that the lilypads cover is 108π square feet. Thus, because Kelvin the Frog's home has total area of $6\cdot8$ square yards, or $48\cdot9$ square feet, the uncovered area is $432-108\pi$ square feet. So, the answer is $432+108=\boxed{540}$.

15. The only positive integers that have an odd number of divisors are perfect squares. Since Rosie's favorite number is less than 100, this gives us 9 possible numbers: $1, 4, 9, 16, 25, 36, 49, 64, 81$. We see that $\boxed{25}$ must be Rosie's favorite number because it is the only number in this set that has a units digit of 5. All the other numbers do not have a unique units digit, so Jennie would not have been able to figure out Rosie's favorite number.

16. First, we consider performances that are hilarious. This implies every judge rated the performance either a 7, 8, 9, or 10. There are three judges, so the lower bound for a hilarious performance is a total score of $7 \cdot 3 = 21$, whereas the upper bound is $10 \cdot 3 = 30$. There are $30 - 21 + 1 = 10$ possible scores for hilarious performances.

    For a performance to be embarrassing, each judge must have rated it either a 1, 2, or 3. The lower bound here is a total score of $1 \cdot 3 = 3$, whereas the upper bound is $3 \cdot 3 = 9$. There are $9 - 3 + 1 = 7$ possible scores for embarrassing performances.

    (Note that all the scores in the ranges are attainable because the total score increases by 1 each time a judge increases his or her score by 1.)

    In total, there are $10 + 7 = \boxed{17}$ possible scores for performances that are either hilarious or embarrassing.

17. To make the numbers cleaner, let the potato farm have an area of 3400. A clone of Jaiden can harvest 2 units per day while a clone of Lance can harvest 1 unit per day so we get the equation $2x + y = 3400$ for $x, y, \geq 0$. This has $\boxed{1701}$ solutions, since x can take any integer value from 0 to 1700.

18. The total area of the cookie is $\pi \cdot 10^2 = 100\pi$. Since a quarter of the area is covered by chocolate chips, the total area of the chips must be 25π. The area of each chip is $\pi \cdot 1^2 = \pi$, so there are $\boxed{25}$ chips on the cookie.

19. In order to guarantee that Harry has picked out a pair of red socks, Harry must consider the worst possible scenario, which is that Harry picks out every blue sock and every green sock before he picks even a single red sock. If Harry picks every green sock and every blue sock, he would only have red socks to choose from after that. So, he would only have to pick 2 more socks to ensure that he has picked at least one pair of red socks. In total, this would be $12 + 15 + 2 = \boxed{29}$ socks.

20. We can connect the center of the rectangle (also the center of the circle) to a vertex of the rectangle (which lies on the circle) to get that half the diagonal of the rectangle is equal to 20, the radius of the circle. From here, we can use the Pythagorean Theorem to get that half the other side of the rectangle is equal to 12, so the entire rectangle has dimensions 32×24 which yields 768 square miles. The original circle was 400π square miles, so $n + m = 400 + 768 = \boxed{1168}$.

21. Of the 79 positive integers less than 80, we can see that 15 are divisible by 5 and 11 are divisible by 7. However, two numbers (35 and 70) are divisible by both and are counted twice, so there are $79-15-11+2 = \boxed{55}$ valid integers.

22. Call Maui's distance to his shore x, and his distance to the cave y. Additionally, call his running speed s_r and his flying speed s_f. Since time is equal to distance divided by speed, $\frac{x}{s_r} = \frac{y}{s_r} + \frac{x+y}{s_f}$. Since we know $s_f = 4s_r$, this equation can be rewritten as $\frac{x}{s_r} = \frac{y}{s_r} + \frac{x+y}{4s_r}$. Finally we can solve the equation:

$$\begin{aligned} \frac{x}{s_r} &= \frac{y}{s_r} + \frac{x+y}{4s_r} \\ 4x &= 4y + x + y \\ 3x &= 5y \\ \implies x : y &= 5 : 3 \end{aligned}$$

    Therefore, $a = 5$ and $b = 3$, so $a + b = \boxed{8}$.

23. We know that each alien is named by <u>abc</u>, where a, b, c are letters, not necessarily distinct. If we are allowed to select 3 letters from n total letters, the maximum number of names this could create would be n^3. Therefore, we must find the value of n such that $n^3 \geq 10,000$. The smallest value of n is $22^3 = 10,648$, which means our answer is $\boxed{22}$.

24. Since we want the number to be as small as possible, a must have the smallest possible value, so let $a = 1$. Then the next smallest number is 2, so let $b = 2$. The only possible digit smaller than two is now 0, so $c = 0$. Then, we take the two smallest remaining numbers to assign to d, e. Since we know $d > e$, $d = 4$ and $e = 3$. We get a result of $\boxed{12043}$.

25. Since the corresponding side lengths of the two triangles have the same ratio, the ratio of their perimeters must also equal this ratio of corresponding sides. This unknown ratio can be 3 values depending on which side of the first triangle corresponds with the 12 cm side of the second triangle:

$$\frac{2}{12} = \frac{1}{6}$$
$$\frac{4}{12} = \frac{1}{3}$$
$$\frac{3}{12} = \frac{1}{4}.$$

We know that this ratio must equal $\frac{2+4+3}{P} = \frac{9}{P}$ where P is the perimeter of the second triangle. Using the three possible ratios found before, we find $P = 54, 27$, or 36. Hence, our answer is $54+27+36 = \boxed{117}$.

26. This can be written as $(3 \cdot 2 \cdot 1) \cdot (5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) \cdot (8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) = 2^{11} \cdot 3^4 \cdot 5^2 \cdot 7$. Then a perfect cube that divides this must be in the form of $2^a \cdot 3^b \cdot 5^c \cdot 7^d$ where a, b, c, d must be multiples of 3, i.e.

$$a \in \{0, 3, 6, 9\}, b \in \{0, 3\}, c \in \{0\}, d \in \{0\}.$$

Thus, there are $4 \cdot 2 \cdot 1 \cdot 1 = \boxed{8}$ perfect cubes that divide the given expression.

27. If we work backwards from 101, we can only subtract 1 and divide by 2. Note that to get to our original number, it is much more efficient to divide by 2. Since 101 is an odd number, we can only subtract 1. This gives 100, and we can now divide by 2 twice, to get 25. Because we again have an odd number, we must subtract 1 to get 24, which means we can divide by 2 three times again. This yields 3, which is what we want. Therefore, we performed the moves, $(3 \cdot 2 \cdot 2 \cdot 2 + 1) \cdot 2 \cdot 2 + 1$, and our answer is $\boxed{7}$.

28. Notice that when the altitude to the side of length 40 cm is drawn in the original triangle, then it is split into two separate ones of side length 20 cm, 21 cm, and 29 cm. That means another triangle can be formed but instead of 21 cm side length being the height, the height is 20 cm. Therefore, the new triangle has side lengths 29 cm, 29 cm, and 42 cm. That means $x = \boxed{42}$.

29. Recall that the divisibility rule for 4 is that if the last two digits of a number are divisible by 4, the whole number is divisible by 4. This means there are $100/4 = 25$ possibilities for the last two digits. As our number is a palindrome, the last digit is the same as the first digit, and the second to last digit is the same as the second digit. Since numbers cannot have leading 0s, the first digit and therefore the last digit cannot be a 0, removing 00, 20, 40, 60, and 80 from the possibilities for the last two digits. The central digit can be anything, so in total we have $(25 - 5) * 10 = \boxed{200}$ valid palindromes.

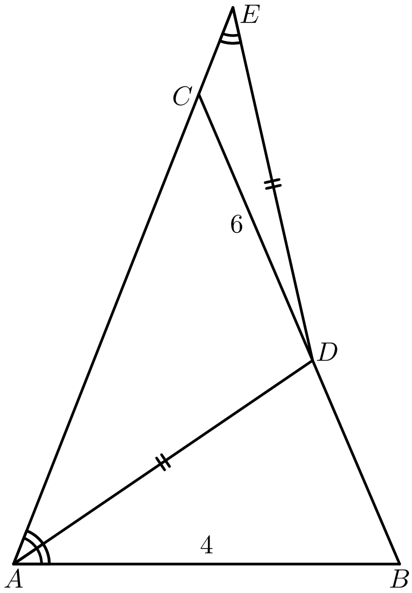
30. We can draw a 2×2 grid, where the rows are 1 scoop and 2 scoop, and the columns are no toppings and 1 topping. 2 kids got 2 scoops with no toppings, so we can fill in that square with a 2. Since 12 kids got 2 scoops, the 2 scoops row must add to 12 so the 2 scoops with toppings square must be 10. Similarly, since 4 kids got no toppings, there must be 2 kids who got 2 scoops with no toppings. Finally, since all squares must add to 20, we have the answer is $20 - 2 - 2 - 10 = \boxed{6}$.

31. First, write the first 21 positive integers. The numbers $(1, 2, 4, 8, 16)$ are all connected and have 2 different colorings: R, B, R, B, R or B, R, B, R, B. The same is true for the groups $(3, 6, 12), (5, 10, 20), (7, 14)$, and $(9, 18)$. The only numbers that are independent are 11, 13, 15, 17, 19, and 21 and they each have two choices R or B. Lastly, doing the calculation gives $2^5 \cdot 2^6 = \boxed{2048}$ different colorings.

32. $\angle DAC$ and $\angle DEC$ are congruent because $\triangle ADE$ is isosceles. Then, $\angle BAD$ is congruent to $\angle CAD$, from which you can then use the Angle Bisector Theorem to get that $\frac{4}{BD} = \frac{AC}{6}$. Thus, $BD \cdot AC = \boxed{24}$.

33. Plug in $(x-a)$ and $(y-b)$ in x and y respectively. Then we get $y-b = 3(x-a)^2+2(x-a)-1$. Expanding and simplifying this, the expression becomes $y = 3x^2-6ax+3a^2+2x-2a-1+b = 3x^2-(6a-2)x+3a^2-2a-1+b$. Then $6a-2=10$ and $3a^2-2a-1+b=12$. We get $a=2$ and $b=5 \implies 2+5=\boxed{7}$.

34. Using geometric probability, a graph can be made mapping the situations where Jasmine and Aladdin's will meet each other. For any time one of them arrives, the other has a 60 minute window to show up (30 minutes before or after). The graph will look like the one below. The center part is the chance where they will meet. Out of the whole graph, that section makes up $\frac{11}{36}$ of it. So $36-11=\boxed{25}$.



(Not to scale)

35. We first consider the number of ways to place Annie and Blake. We use complementary counting, subtracting the number of arrangements where Annie and Blake are adjacent from the total number of arrangements. There are $5 \cdot 4 = 20$ total ways to place them; Annie can stand in any of the five positions, and Blake can stand in any of the remaining four. There are $4 \cdot 2 = 8$ invalid ways they can be put in line; there are four spaces where the person in front can stand, and 2 choices for the person in front. Thus, there are 12 valid ways Annie and Blake can be placed.

    Disregarding the requirement that Eric stand in front of Dong Joo, there are $3 \cdot 2 \cdot 1 = 6$ ways to place Christina, Dong Joo, and Eric once Annie and Blake are in line. By symmetry, Eric will be standing in front of Dong Joo in exactly half of these cases, so for each arrangement of Annie and Blake, there are 3 acceptable placements of Christina, Dong Joo, and Eric. Therefore, there are $12 \cdot 3 = \boxed{36}$ ways for the five to stand in line.

36. Consider $g(x) = f(x) - x$, so the only roots of $g(x)$ are $x=1$ and $x=2$. This means g must be either $g(x) = (x-1)^2(x-2) \implies g(0) = -2$ or $g(x) = (x-1)(x-2)^2 \implies g(0) = -4$. Because $f(x) = g(x) + x \implies f(0) = g(0)$, the answer is $-2 + -4 = \boxed{-6}$.

37. Draw diagonal ML to divide $YULKIM$ into two isosceles trapezoids. Then drop perpendiculars from Y, U, K, and I onto ML. From this, use the 30-60-90 triangles to find the area of the whole figure. If we denote the side lengths a, b, then the area is $(ab + \frac{a^2+b^2}{4})\sqrt{3}$. Plugging in $a=4, b=6$, this evaluates to $37\sqrt{3} \implies [YULKIM]^2 = 37^2 \cdot 3 = \boxed{4107}$.

38. Let the number of blue socks be x. Therefore, we have $x+10$ red socks and $2x+10$ in total. We also know that the probability of Bobette getting a pair of same-colored socks is $\frac{1}{2}$. The number of ways to get a pair of red socks is $(x+10)(x+9)$ and the number of ways to get a pair of blue socks is $(x)(x-1)$. Furthermore, the total number of ways to choose two socks is $(2x+10)(2x+9)$. We can now write

$$\frac{(x+10)(x+9)+(x)(x-1)}{(2x+10)(2x+9) = \frac{1}{2}}.$$

    Solving for x we get 45. Therefore, Bobette has $\boxed{45}$ blue socks.

39. By inspection, $3^3 = 27 \equiv 6 \pmod 7$ and $3^6 \equiv 6^2 \equiv 1 \pmod 7$. This cycle repeats as $3^7 = 3^6 * 3 \equiv 3 \pmod 7$.

    We note that the exponent of $3^{9^{27^{81}}}$, $9^{27^{81}}$, is an odd multiple of 3, so it can be written as $6k+3$ for some non-negative integer k. Hence, $3^{9^{27^{81}}} = 3^{6k+3} = (3^6)^k \cdot 3^3 \equiv 6 \pmod 7$. Thus, the remainder is $\boxed{6}$.

40. n_i is essentially the base-3 representation of i, but substituting the 0s with 1s, the 1s with 3s, and the 2s with 9s (try to convince yourself why!) The base-3 representation of 2021 is 2202212, which results in $a_n = \boxed{9919939}$.