Joe Holbrook Memorial Math Competition

6th Grade Solutions

October 17, 2021

1. Let x be the number of windows on her Mac. Then she has $3x$ windows on her PC. In total, she has $x + 3x = 4x$ computers, so Bianca has $4x = 20 \implies x = 5 \implies 3x = \boxed{15}$ windows on her PC.

2. There are 90 - 45 - 30 = 15 blue beads, so the ratio of blue beads to red beads is 15:45 = 1:3 $\implies a+b = \boxed{4}$

3. We want $4 \cdot 5 = 2 \cdot x$, where x is the side length of the rectangle. Solving, we get that $x = \boxed{10}$.

4. Let m be Maryam's age and y be Yuval's age. From the problem, we know that

$$m - 5 = 2(y - 5)$$

and

$$m - 11 = 3(y - 11).$$

From here, we know

$$m = 2(y - 5) + 5 = 3(y - 11) + 11$$
$$2y - 5 = 3y - 22$$
$$y = 17,$$

so Yuval is $\boxed{17}$ years old.

5. In essence, we are trying to find the diameter of the circular park, as they are walking the path created from the opposite ends of the circle. Because both friends walked 18π miles, they walked 36π miles in total, implying that the circumference of the circle is 36π. The circumference of a circle is equal to its diameter times π, so the diameter is is $\boxed{36}$.

6. The square has side length $\sqrt{16} = 4$. Thus, the pentagon has 5 sides of this length, so the perimeter is $5 \cdot 4 = \boxed{20}$.

7. Since Cookie Monster's cookie costs 67 cents, we know he must use the 2 pennies to pay. Also, because we have the condition that he must pay using at least 3 dimes, we can subtract that out so that we do not have any more restrictions left. Now, we need to find the number of ways to pay for 35 cents using at max 3 pennies, 5 nickels, 2 dimes, and 5 quarters. This can be done with 2 nickels + 1 quarter, or 1 dime + 1 quarter, 3 nickels + 2 dimes, or 5 nickels + 1 dimes. This yields the answer of $\boxed{4}$.

8. We start by finding L. Lili's list is $6, 12, \ldots, 96$ which can be rewritten as $1 \cdot 6, 2 \cdot 6, \ldots 16 \cdot 6$. Clearly, $L = 16$. Zoey's list is $1, 5, \ldots, 97$ which can be rewritten as $0 \cdot 4 + 1, 1 \cdot 4 + 1, ..., 24 \cdot 4 + 1$. Clearly $Z = 25$. Hence our answer is $16 - 25 = \boxed{-9}$

9. We can set up an equation

$$(r + 15) = 4(b + 15)$$

where r = the number of students who own only a red car and b = the number of students who own only a blue car. We can simplify this equation:

$$r + 15 = 4b + 60 \implies r = 4b + 45$$

We also know that Alex asked 100 students, so $r + b + 15 = 100$. If we substitute $4b + 45$ in for r, we get:

$$b + (4b + 45) + 15 = 100 \implies 5b + 60 = 100 \implies 5b = 40 \implies b = 8$$

So, if we take the 8 people who own only a blue car and the 15 people who own both a red and a blue car, we get 8 + 15 = $\boxed{23}$ people who own a blue car.

10. Note that the prime factorization of 25 is $5 \cdot 5$, so the only integers that divide it evenly are 1, 5, -1, and -5. Indeed, $1 \cdot 5 \cdot -1 \cdot -5 = 25$, and $1^2 + 5^2 + (-1)^2 + (-5)^2 = 1 + 25 + 1 + 25 = \boxed{52}$.

11. Let s be the price of a bottle of soda and g be the price of a pack of gum. We are given that $4s \leq 5$ and $8g \leq 5$. From the first inequality, we get that $s \leq 1.25$ which means that a bottle of soda can cost \$1.25 at most. For gum, we get that $g \leq 0.625$, but we know that it most cost a whole number of cents, meaning gum can cost \$0.62 at most. This gives $3(125) + 6(62) = \boxed{747}$ cents as our answer.

12. The only positive integers that have an odd number of divisors are perfect squares. Since Rosie's favorite number is less than 100, this gives us 9 possible numbers: $1, 4, 9, 16, 25, 36, 49, 64, 81$. We see that $\boxed{25}$ must be Rosie's favorite number because it is the only number in this set that has a units digit of 5. All the other numbers do not have a unique units digit, so Jennie would not have been able to figure out Rosie's favorite number.

13. First, we consider performances that are hilarious. This implies every judge rated the performance either a 7, 8, 9, or 10. There are three judges, so the lower bound for a hilarious performance is a total score of $7 \cdot 3 = 21$, whereas the upper bound is $10 \cdot 3 = 30$. There are $30 - 21 + 1 = 10$ possible scores for hilarious performances.

    For a performance to be embarrassing, each judge must have rated it either a 1, 2, or 3. The lower bound here is a total score of $1 \cdot 3 = 3$, whereas the upper bound is $3 \cdot 3 = 9$. There are $9 - 3 + 1 = 7$ possible scores for embarrassing performances.

    (Note that all the scores in the ranges are attainable because the total score increases by 1 each time a judge increases his or her score by 1.)

    In total, there are $10 + 7 = \boxed{17}$ possible scores for performances that are either hilarious or embarrassing.

14. To make the numbers cleaner, let the potato farm have an area of 3400. A clone of Jaiden can harvest 2 units per day while a clone of Lance can harvest 1 unit per day so we get the equation $2x + y = 3400$ for $x, y, \geq 0$. This has $\boxed{1701}$ solutions, since x can take any integer value from 0 to 1700.

15. The total area of the cookie is $\pi \cdot 10^2 = 100\pi$. Since a quarter of the area is covered by chocolate chips, the total area of the chips must be 25π. The area of each chip is $\pi \cdot 1^2 = \pi$, so there are $\boxed{25}$ chips on the cookie.

16. We can connect the center of the rectangle (also the center of the circle) to a vertex of the rectangle (which lies on the circle) to get that half the diagonal of the rectangle is equal to 20, the radius of the circle. From here, we can use the Pythagorean Theorem to get that half the other side of the rectangle is equal to 12, so the entire rectangle has dimensions 32×24 which yields 768 square miles. The original circle was 400π square miles, so $n + m = 400 + 768 = \boxed{1168}$.

17. Call Maui's distance to his boat x, and his distance to the cave y. Additionally, call his running speed s_r and his flying speed s_f. Since time is equal to distance divided by speed, $\frac{x}{s_r} = \frac{y}{s_r} + \frac{x+y}{s_f}$. Since we know $s_f = 4s_r$, this equation can be rewritten as $\frac{x}{s_r} = \frac{y}{s_r} + \frac{x+y}{4s_r}$. Finally we can solve the equation:

$$\begin{aligned} \frac{x}{s_r} &= \frac{y}{s_r} + \frac{x+y}{4s_r} \\ 4x &= 4y + x + y \\ 3x &= 5y \\ \implies x : y &= 5 : 3 \end{aligned}$$

    Therefore, $a = 5$ and $b = 3$, so $a + b = \boxed{8}$.

18. If the repeated four-digit integer is n, then the eight-digit integer is $10^4 n + n = 10001n$. That means the population can at least be split into groups of 10001 people. Since the GCD of all possible values of n is 1, we know for a fact that $\boxed{10001}$ is the largest group that can be formed.

19. There will be no real solution for x when the discriminant, $b^2 - 4ac$ is less than 0. For $f(x)$, this yields the inequality $k^2 - 4(100) < 0$. This means that $k^2 < 400$, so $-20 < k < 20$. Because this interval for k gives 19 negative and 19 positive solutions, along with $k = 0$, the answer is $\boxed{39}$.

20. Since the corresponding side lengths of the two triangles have the same ratio, the ratio of their perimeters must also equal this ratio of corresponding sides. This unknown ratio can be 3 values depending on which side of the first triangle corresponds with the 12 cm side of the second triangle:

$$\frac{2}{12} = \frac{1}{6}$$
$$\frac{4}{12} = \frac{1}{3}$$
$$\frac{3}{12} = \frac{1}{4}.$$

We know that this ratio must equal $\frac{2+4+3}{P} = \frac{9}{P}$ where P is the perimeter of the second triangle. Using the three possible ratios found before, we find $P = 54, 27$, or 36. Hence, our answer is $54+27+36 = \boxed{117}$.

21. If we work backwards from 101, we can only subtract 1 and divide by 2. Note that to get to our original number, it is much more efficient to divide by 2. Since 101 is an odd number, we can only subtract 1. This gives 100, and we can now divide by 2 twice, to get 25. Because we again have an odd number, we must subtract 1 to get 24, which means we can divide by 2 three times again. This yields 3, which is what we want. Therefore, we performed the moves, $(3 \cdot 2 \cdot 2 \cdot 2 + 1) \cdot 2 \cdot 2 + 1$, and our answer is $\boxed{7}$.

22. Letting $x = 12$ and $k = \frac{23}{12}$, we have that $f(23) = f\left(\frac{23}{12} \cdot 12\right) = \frac{23}{12} \cdot f(12) = \frac{23}{12} \cdot 108 = 23 \cdot 9 = \boxed{207}$. As a note, the function $f(x) = 9x$ is a function that satisfies $kf(x) = f(kx), f(12) = 108$).

23. Since the numbers are multiples of five, their unit's digits must be 0 or 5. We proceed with casework on these two possibilities.

Case 1: The unit's digit is 5. From here, we split into two more sub-cases: 5 is the repeated digit and 5 is not the repeated digit.

- Case 1.1: 5 is the repeated digit. The number must take the form $5X5$ or $X55$. For the first form, the missing digit can be any digit except for 5 and for the second form, the missing digit can be any digit except for 5 and 0. This gives us $9 + 8 = 17$ for this sub-case.
- Case 1.2: 5 is not the repeated digit. The number must take the form $XX5$. The repeated digit can be any digit except for 0 so we have 9 numbers for this sub-case.

Case 2: The unit's digit is 0. Again, we split into two more sub-cases: 0 is the repeated digit and 0 is not the repeated digit.

- Case 2.1: 0 is the repeated digit. The number must take the form $X00$, giving us 9 numbers.
- Case 2.2: 0 is not the repeated digit. The number must take the form $XX0$, giving us 9 numbers.

In total, we have $17 + 9 + 9 + 9 = \boxed{44}$ lucky numbers.

24. Recall that the divisibility rule for 4 is that if the last two digits of a number are divisible by 4, the whole number is divisible by 4. This means there are $\frac{100}{4} = 25$ possibilities for the last two digits. As our number is a palindrome, the last digit is the same as the first digit, and the second to last digit is the same as the second digit. Since numbers cannot have leading 0s, the first digit and therefore the last digit cannot be a 0, removing 00, 20, 40, 60, and 80 from the possibilities for the last two digits. The central digit can be anything, so in total we have $(25 - 5) * 10 = \boxed{200}$ valid palindromes.

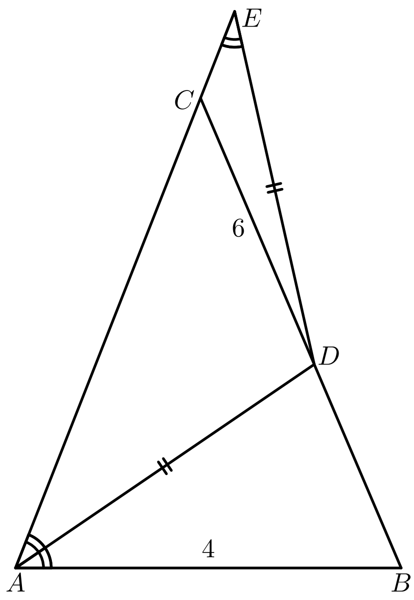
25. Squaring both sides of the equation, we have $x + 5\sqrt{x + 5\sqrt{x + 5\sqrt{x + 5 + \cdots}}} = 100$. However, since we know $\sqrt{x + 5\sqrt{x + 5\sqrt{x + 5 \cdots}}} = 10$, we have that $x + 5(10) = 100$, so $x = 100 - 50 = \boxed{50}$.

26. Since $(5 + |x - 10|)^2 = 100$, we know that $5 + |x - 10| = \pm 10$. First we solve the equation $5 + |x - 10| = 10$. This gives us $|x - 10| = 5$. Then we know $x - 10 = \pm 5$, which gives us the solutions $x = 5, 15$.

Next if $5 + |x - 10| = -10$, then $|x - 10| = -15$. This is not possible because the absolute value of a number is always nonnegative. Thus this has no solutions, so our only solutions are $x = 5, 15$. The product of these is $5 \times 15 = \boxed{75}$.

27. The total number of ways of choosing three cards is $\binom{5}{3} = 10$. The number formed by the last two digits of any multiple of 4 will also be a multiple of 4. Also, a multiple of 4 must contain an even digit so we divide into cases where it contains a 2 or a 4. If 2 is one of the three numbers, the three cards can always be arranged to be a number divisible by 4. This is because at least one of 1,3, or 5 must be one of the three cards and 12, 32, 52 are all multiples of 4. There are 6 cases where 2 is included in the three numbers. Then, there are 3 cases where 4 is included but 2 is not: $(1,3,4),(1,4,5),(3,4,5)$. None of these can be arranged to be a multiple of 4. Finally, $(1,3,5)$ clearly cannot be arranged to be a multiple of 4. Therefore, the possibility is $\frac{6}{10} = \frac{3}{5}$ and the answer is $3+5 = \boxed{8}$.

28. We know $\angle DAC$ and $\angle DEC$ are congruent because $\triangle ADE$ is isosceles. Then, $\angle BAD$ is congruent to $\angle CAD$, from which you can then use the Angle Bisector Theorem to get that $\frac{4}{BD} = \frac{AC}{6}$. Thus, $BD \cdot AC = \boxed{24}$.



(Not to scale)

29. Choose F on AB such that EF is perpendicular to AB, then extend EF to meet CD at G. Notice $\triangle EDG$ is a right triangle with hypotenuse DE. We know $DG = \frac{1}{2}, FG = 1, EF = \frac{1}{2}$. (The last one might be harder to see, but $\triangle AEF$ and $\triangle BEF$ are both also isosceles right triangles.) By the Pythagorean Theorem, $DE = \sqrt{(\frac{1}{2})^2 + (1+\frac{1}{2})^2} = \sqrt{\frac{10}{4}} = \sqrt{\frac{5}{2}} \implies a+b = 5+2 = \boxed{7}$.

30. Let $P = 4, 8, 12, \cdots, 112, 116$ be the set of points that divide the log into 30 pieces. Similarly, let $Q = 6, 12, 18, \cdots 108, 114$ be the set of points that divide the log into 20 pieces and let $R = 10, 20, 30, ..., 100, 110$ be the set of points that divide the log into 12 pieces. Now, we can use the Principle of Inclusion-Exclusion (PIE). First, we add the number of elements in the three sets: 29+19+11=59. Then we subtract the number of elements in each $P \cap Q, Q \cap R$, and $R \cap P$: $59 - 9 - 3 - 5 = 42$. (Note: $P \cap Q$ denotes the set of elements in either P or Q). Finally we add back the number of elements in $P \cap Q \cap R$ which is 1, so we get $42 + 1 = 43$ marks and $\boxed{44}$ pieces.

31. We know that $F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8, F_7 = 13, F_8 = 21, F_9 = 34, F_{10} = 55, F_{11} = 89, F_{12} = 144, F_{13} = 233, F_{14} = 377, F_{15} = 610$. In order for three numbers to multiply to a multiple of 10, there must be a factor of 2 and 5. The factor of 2 can come from F_3, F_6, F_9, F_{12}, or F_{15}, while the factor of 5 can come from F_5, F_{10}, or F_{15}. Since $F_{15} = 610$ is a multiple of 10 on its own, we can do casework based on it.

   - Case I. - F_{15} is a chosen number. This means that we can choose pair of remaining numbers, which yields $\binom{14}{2} = 91$. However, the numbers must be distinct, so the case of $\{F_1, F_2, F_15\} = \{1, 1, 610\}$ should be subtracted, giving 90 triples for Case I.
   - Case II. - F_{15} is not a chosen number. Then, we must pick a multiple of 2, a multiple of 5, and another number. If we only pick multiples of 2's and 5's, this will give us $2\binom{4}{2} + 4\binom{2}{2} = 16$. If we pick a multiple of 2, a multiple of 5, and another non-multiple of 2 and 5, it is $4 \cdot 2 \cdot 8 = 64$. Thus, our answer is
   $$90 + 16 + 64 = \boxed{170}.$$

32. If Jerome wins, then the last coin flip must be tails, so we disregard this flip. The previous flips include 3 tails and up to 2 heads. If no heads are rolled, then the flip sequence must be TTT, which has a probability of $\frac{1}{2^3} = \frac{1}{8}$ of occurring. If 1 head is rolled, then the flip sequence must be 3 tails and 1 head in some order. There are $\binom{4}{3} = 4$ ways for this to occur, and a probability of $\frac{4}{2^4} = \frac{1}{4}$. If 2 heads are rolled, then the flip sequence has 3 tails and 2 heads, which can occur in $\binom{5}{3} = 10$ ways and has a probability of $\frac{10}{2^5} = \frac{5}{16}$ of occurring. Summing these probabilities, we get $\frac{1}{8} + \frac{1}{4} + \frac{5}{16} = \frac{11}{16}$. However, we have to divide by 2 because the last flip is T in only half of these cases, so we get $11 + 32 = \boxed{43}$.

33. Let each $x_i = 2k_i$. Now, we have that $2(k_1 + k_2 + k_3) = 56$, so $k_1 + k_2 + k_3 = 28$ where each k_i is nonnegative. We can now use the technique of stars and bars (look this up if you don't know what it is!) to see that

$$n = \binom{28+3-1}{3-1} = \binom{30}{2} = \boxed{435}$$

34. If the gcd is something other than 1, it must be 2 or 3 (there don't exist 3 multiples of 4 less than 10). The multiples of 2 from 1 to 10 are 2, 4, 6, 8, and 10, yielding $\binom{5}{3} = 10$ possibilities. The multiples of 3 from 1 to 10 are 3, 6, and 9, yielding $\binom{3}{3} = 1$ possibilities. There are $\binom{10}{3} = 120$ triples in total, so the answer is $120 - 11 = \boxed{109}$.

35. Let A, E, L, Y, D represent the amount of candy Autumn, Erez, Alicia, Yul, and David receive, respectively. Let $Y' = Y - 5$, so that $A, E, L, Y', D \geq 0$ with no other restrictions. This will allow us to use stars and bars. We can now conduct casework on the value of A:

- $A = 0$: $E + L + Y' + D = 8$. By stars and bars, there are $\binom{11}{3} = 165$ ways.
- $A = 1$: $E + L + Y' + D = 7 \implies \binom{10}{3} = 120$ ways.
- $A = 2$: $E + L + Y' + D = 6 \implies \binom{9}{3} = 84$ ways.

This sums to a total $165 + 120 + 84 = \boxed{369}$ ways.

36. The graph of $x^2 \leq (2-y)(2+y) \implies x^2 + y^2 \leq 4$ is a circle centered at the origin with radius 2, while the graph of $|x| + |y| \leq 2|$ is a square of length $2\sqrt{2}$, centered at the origin and rotated $45°$.The region is a circle with the inscribed square cut off, so the answer is $4\pi - 8$ and the final sum is $\boxed{-4}$.

37. Let's analyze the first few powers of 3 when divided by 7. We see that

$$\begin{aligned} 3^0 &\equiv 1 \pmod 7 \\ 3^1 &\equiv 3 \pmod 7 \\ 3^2 &\equiv 2 \pmod 7 \\ 3^3 &\equiv 6 \pmod 7 \\ 3^4 &\equiv 4 \pmod 7 \\ 3^5 &\equiv 5 \pmod 7 \\ 3^6 \equiv 3^0 &\equiv 1 \pmod 7 \end{aligned}$$

This means that the powers of 3 repeat in cycles of 6 $\pmod 7$, so $1+3+3^2+\cdots+3^{200} \equiv (1+3+2+6+4+5)+(1+3+2+6+4+5)+\cdots+(1+3+2+6+4+5)+(1+3+2) \equiv 33\cdot(1+3+2+6+4+5)+(1+3+2) \equiv 699 \pmod 7 \equiv \boxed{6} \pmod 7$.

38. Clearly, we can get the area of ABC as close to 0 as we want, by making BC close to 0 and AB close to 10. However, if $[ABC] = 0$, then that wouldn't be a proper triangle, so the smallest possible integer area is 1.

For an upper bound, consider all possible triangles with a fixed BC. Dropping a height from point B onto CA, this height is always less than BC by the Pythagorean Theorem, unless $\angle BCA = 90°$, and the height is equal to BC. Then, the area is $10 \cdot BC/2 = 5BC$. We can get BC as close to 10 as we want, which would result in an area of 50. Since we can't actually get 50, the maximum is 49. Thus, there are $\boxed{49}$ possible values, from 1 to 49.

39. There are $\binom{11}{5}$ ways to place the five characters of "lance" in the 11-letter string. However, we overcount some cases, since there are 2 n's and l's. The first case is when the two n's are next to each other, which happens in

lann _ _ _ _ _ _ _

There are $\binom{7}{2}$ ways to place the rest of the characters. When the l's are next to each other, this only happens one way: nikhillance. So, our answer is

$$\binom{11}{5} - \binom{7}{2} - 1 = \boxed{440}$$