Joe Holbrook Memorial Math Competition

7th Grade Solutions

October 17, 2021

1. Alice computes $6 \div 3 = 2$ first, then $9 + 2 = 11$. Bob computes $9 + 6 = 15$, then $15 \div 3 = 5$. The difference is $11 - 5 = \boxed{6}$.

2. Suppose the book has n pages. Because she finishes the book, we know that
$$\frac{1}{4}n + \frac{1}{3}n + 55 = n.$$
Simplifying, we see
$$\frac{5}{12}n = 55$$
so $n = 55 \cdot \frac{12}{5} = \boxed{132}$ pages.

3. Bob needs to choose the wrapping paper color, the box shape, and the ribbon type. This can be done in $4 \cdot 5 \cdot 2 = \boxed{40}$.

4. To ensure that Rufus barks the maximum number of times in a 18 hour period, he must begin at 7 o'clock and end at 12 o'clock after the 18 hours have passed. This yields the sum
$$7 + 8 + 9 + \cdots + 12 + 1 + 2 + 3 + \cdots + 12 = \boxed{135}$$

5. Set up two equations, $9x + 5y = 740$ and $x + y = 120$, where x is Ferris wheel tickets and y is roller coaster tickets.brybe Multiply the second equation by 5 to get $5x + 5y = 600$. Then subtract the equations to cancel the y variable. This gives $4x = 140$. Divide by 4 on both sides to get that $\boxed{x = 35}$.

6. Notice that for the time to be a palindrome, the minute is determined by the hour. Therefore, the next palindrome must be after 1:00, as 12:21 is the only palindrome between 12:00 and 1:00. Then the next palindrome must be 1:01 a.m., and this is $\boxed{40}$ minutes after 12:21.

7. Clearly, one of the groups of apples must have $4 \cdot 5$, $5 \cdot 7$, or $4 \cdot 7$ apples. The last two are greater than 27, so Jaiden splits the apples into 20 and 7. Since Lance gets more than half, Lance gets $\boxed{20}$.

8. Let $S = 1 + 2 - 3 + 4 + 5 - 6 + \cdots + 97 + 98 - 99$. Evaluate these sums in groups of 3.
$$S = (1 + 2 - 3) + (4 + 5 - 6) + \cdots + (97 + 98 - 99) = 0 + 3 + 6 + \cdots + 96$$
We can evaluate this sum using the arithmetic series formula:
$$S = \frac{n(a_n - a_1)}{2} = \frac{33(96)}{2} = \boxed{1584}$$

9. Notice the triangle is a right triangle (3-4-5 triangle scaled up by 2) with legs of length 6 and 8, so its area is 24. An equilateral octagon has 8 sides of equal length, so if the perimeter is 24, each side must have length $24 \div 8 = \boxed{3}$.

10. The prime factorization of 105 gives 3,5, and 7. The product of any pair of these gives a result greater than 9, which would not be a digit. Therefore 3 of the digits are 3, 5, and 7, and the rest are 1s. The sum is $3 + 5 + 7 + 7 \cdot 1 = \boxed{22}$.

11. Note that the letter e must be in the middle as it is the only letter without a pair. Furthermore, there must be one r, one a, and one c on each side. The arrangement of letters on the left side of the e will determine the right side, since the right will just be the reverse of the left. Therefore, only the three leftmost slots need to be considered. There are $3 \cdot 2 \cdot 1 = \boxed{6}$ ways to arrange the letters in the three slots.

12. After 154 seconds, a Pifan will have said $2 \cdot 154 = 308$ digits. This means that they have fully repeated the string of 15 digits 20 times, along with the first 8 digits of the string. Since the sum of a full string is $3+1+4+1+5+9+2+6+5+3+5+8+9+7+9 = 77$, then 20 times would yield a total sum of 1540. However, we still have 8 digits remaining, of which their sum is $3+1+4+1+5+9+2+6 = 31$. Our answer is $1540 + 31 = \boxed{1571}$.

13. The probability of rolling no 20s is the same whether or not Nikhil rolls a pair of 20s before the proof roll. The probability of getting a not-20 on one die is $\frac{19}{20}$, so the probability of getting a not-20 on both dice is $\frac{19}{20} \cdot \frac{19}{20} = \frac{361}{400}$, so the answer is $\boxed{761}$.

14. Because 25 hats for \$10 is the best deal, and he needs more than 25, he should take this deal first; he should then take the 10 for \$5 deal, since paying another \$10 for 25 is unnecessary (he only needs 38 hats). After doing so, he only needs to get 3 more party hats, which he can use the 1 for \$1 deal to get. This adds up to $\$10 + \$5 + \$3 = \boxed{\$18}$.

15. There are $4 \cdot 4 \cdot 4 = 64$ possible ways to assign a teacher to Autumn, David, and Erez. There are four different teachers that the three students could all be assigned to, so 4 of the 64 assignments allow the students to have the same teacher. This yields a probability of $\frac{4}{64} = \frac{1}{16}$, which gives us an answer of $1 + 16 = \boxed{17}$.

16. *Good News* playing right after *Weekend* is only possible when *Weekend* is one of the first 16 songs played (which has probability $\frac{16}{17}$), and *Good News* has a $\frac{1}{16}$ chance to be the song that plays right after *Weekend.* $\frac{16}{17} \cdot \frac{1}{16} = \frac{1}{17} \implies \boxed{18}$.

17. Notice that the paths of Alicia and Yul form a right triangle with legs 15 feet and 20 feet, where Alicia walks the legs and Yul walks the hypotenuse. We can notice that this is a 3-4-5 triangle to find that the hypotenuse is 25 feet. Thus, Alicia must walk $(15 + 20) - 25 = \boxed{10}$ more feet than Yul.

18. Let the vertices of the square be A, B, C, D, the center of the square be O, and the vertex joining the 4 equilateral triangles be E. Through Pythagorean theorem, we can show that $\overline{AO} = \sqrt{2}$ Now, since $\overline{AE} = 2$ because all sides of a equilateral triangle are equal, we can use Pythagorean theorem again to see that the height of the pyramid is $\sqrt{2}$. This is enough to solve for the volume, which is just

$$\frac{2 \cdot 2 \cdot \sqrt{2}}{3} = \frac{4\sqrt{2}}{3}$$

Thus, the answer is $4 + 2 + 3 = \boxed{9}$.

19. We will call the intersection of $\overline{AD}$ and $\overline{MF}$ G, and the intersection of $\overline{BC}$ and $\overline{ME}$ H. The area common to both regions, $[CDGMH]$, is equal to $[ABCD] - [AGM] - [BHM]$. Because $\angle AGM \cong \angle DGF$, and $\angle DFG \cong \angle AMG$, $\triangle FDG \sim \triangle MAG$. Similarly, $\triangle ECH \sim \triangle MBH$.
This implies that $\frac{DG}{DF} = \frac{AG}{AM}$ and that $\frac{CH}{CE} = \frac{BH}{BM}$. Recall that $DF = CE = 6$, that $AM = BM = 3$, and that $AG + DG = BH + CH = 6$. This implies that $AG = 2DG$ and $BH = 2CH$, and therefore $AG = BH = 2$. Thus, $[AGM] = [BHM] = \frac{1}{2} \cdot 3 \cdot 2 = 3$, and finally, $[CDGMH] = 6^2 - 3 - 3 = \boxed{30}$.

20. Let $y = \frac{x}{60}$. Every mile David runs takes him y minutes longer to run than the previous mile. The first mile takes him 8.5 minutes, the second $8.5+y$ minutes, the third $8.5+2y$ minutes, etc. These times sum to $8.5+(8.5+y)+(8.5+2y)+\cdots(8.5+5y) = 8.5\cdot 6+(y+2y+\cdots 5y) = 51+\frac{5\cdot 6}{2}y = 51+15y$ minutes. From the problem statement, we know this is equal to 1 hour, or 60 minutes. $51+15y = 60 \implies y = \frac{3}{5} \implies x = \boxed{36}$ seconds.

21. There are $\binom{12}{6} = \frac{12\times 11\times 10\times 9\times 8\times 7}{6\times 5\times 4\times 3\times 2\times 1} = 924$ ways to make a group of 6 men with the leftover men being the second group. However, since the groups are interchangeable, each group formation is counted twice, so the actual number of ways to split up the 12 men is $\frac{924}{2} = \boxed{462}$.

22. The probability that David does not throw his stuffed turtle at Autumn the first time she pokes him is $1 - \frac{1}{10} = \frac{9}{10}$, and the probability that he also does not throw his stuffed turtle at her the second time she pokes him is $1 - \frac{2}{10} = \frac{8}{10}$. Finally, the probability that he does throw his stuffed turtle at her the third time is $\frac{3}{10}$. This yields a probability of $\frac{9}{10} \cdot \frac{8}{10} \cdot \frac{3}{10} = \frac{27}{125}$ which gives us an answer of $27 + 125 = \boxed{152}$.

23. Let the side length of the original square be 1, so the area of the smaller circle will be the ratio. Notice the diameter of the larger circle is 1, so the radius is $\frac{1}{2}$. Then the radius of the smaller circle can be calculated using an isosceles right triangle with a radius of the large circle as the hypotenuse and a radius of the small circle as a leg. We get $r = \frac{1}{2\sqrt{2}}$, which results in an area of $\frac{\pi}{8} \implies a + b = 1 + 8 = \boxed{9}$.

24. We split this into cases.
Case 1: You buy exactly 1 scoop of cookies-and-cream. Then for the last 2 scoops: you can have 2 different flavors, which is $\binom{3}{2} = 3$ choices, or the same flavor, which is $\binom{3}{1} = 3$ choices. (Or you can simply list out all the combinations!) This is 6 choices in total.
Case 2: You buy exactly 2 scoops of cookies-and-cream. Then there are 3 choices for the 3rd scoop.
Case 3: You buy exactly 3 scoops of cookies-and-cream. There is only 1 choice for this.
Adding all 3 cases, we get a total of $6 + 3 + 1 = \boxed{10}$ orders in total.

25. Bessie can graze on $\frac{3}{4}$ of a 3-radius circle and $\frac{1}{4}$ of a 1-radius circle, while Elsie can graze on $\frac{3}{4}$ of a 1-radius circle. Thus, the area is $\frac{3}{4} \times \pi \times 3^2 + \frac{4}{4} \times \pi \times 1^2 = \frac{31\pi}{4}$ and the answer is $31 + 4 = \boxed{35}$.

26. Since n is divisible by 45, it must be divisible by both 5 and 9. Thus it must end in a 5 or 0, and the sum of its digits must be divisible by 9. Since there are no 5s in n, the last digit must be 0. For the sum of the digits of n to be divisible by 9, the number of 6s in n must be divisible by 3. Thus the smallest possible value of n is 6660, so our answer is $\frac{6660}{45} = \boxed{148}$.

27. Note that $x^2 - 10x + 16$ factors as $(x-2)(x-8)$. As the absolute value of this must be a prime, one of $x-2$ or $x-8$ must be equal to ± 1. If $x - 2 = 1, x = 3$ and $|(x-2)(x-8)| = 5$, a prime. If $x - 2 = -1, x = 1$ and $|(x-2)(x-8)| = 7$, a prime. If $x - 8 = 1, x = 9$ and $|(x-2)(x-8)| = 7$, a prime. If $x - 8 = -1, x = 7$ and $|(x-2)(x-8)| = 5$, a prime. All of these work, so the answer is $3 + 1 + 9 + 7 = \boxed{20}$.

28. Notice that Andrew cannot change his name to become Bandrew, Candrew, or Dandrew since the names have different lengths. If Andrew changes the first letter of his name, he has 24 choices, as he cannot change it to an A or a E. The other five letters can change to anything he wants, besides what they already were. For example, he can change the n to any letter but n, the d to any letter but d, etc. Thus, he has $25 \cdot 5 + 24 = 149$ ways to modify his name.

Similarly, Bandrew has $25 \cdot 6$ ways to modify any of the last 6 letters of his name. He can change the first letter to anything besides B, C, D which leaves 23 choices. This is a total of $25 \cdot 6 + 23 = 173$ possible ways Bandrew can change a letter in his name. Andrew and Bandrew's name modifications are independent so there is a total of

$$149 \cdot 173 = \boxed{25777}$$

possible changes they could make.

29. There are 6 rows/columns and 5 rooks; each rook occupies one row and one column, so after all the rooks are placed there will be one column and one row open. A 6th rook can be placed in that spot without attacking any other rooks. Thus, we can instead consider the number of ways to place 6 rooks on the board and multiply the final answer by 6 (any 1 of the 6 rooks may be removed to make a 5-rook arrangement). To arrange 6 rooks on a chessboard, we can imagine placing the 6 rooks on the diagonal of the board, then rearrange the rows in any way, leading to 6! possible 6-rook arrangements. Each of those arrangements can lead to 6 5-rook arrangements, so the number of 5-rook arrangements is $6 \cdot 6! = \boxed{4320}$.

30. Since $a_2 = a_1 r$, and using the fact that $a \cdot b = \text{lcm}(a, b) \gcd(a, b)$, we have that

$$a_1 r = \text{lcm}(a_1, r) \cdot \gcd(a_1, r) = \text{lcm}(a_1, r)^2$$

which gives that $\gcd(a_1, r) = \text{lcm}(a_1, r)$. This implies that $a = r$ so $a_{17} = ar^{16} = 17^{34} \implies a = r = 289$, so our answer is $\boxed{578}$.

31. Dory is basically counting in a form of base 9 where when she counts $1, 2, 4, 5, 6, 7, 8, 9$ she is counting the base 9 equivalent of $1, 2, 3, 4, 5, 6, 7, 8$. Therefore, the number 456 in Dory's counting method is like the number 345_9. Now, to get the real number of fish just convert 345_9 to base 10. $345_9 = 9^2 \times 3 + 9^1 \times 4 + 9^0 \times 5 = 284_{10}$. Therefore, there are actually $\boxed{284}$ fish in the coral reef.

32. This problem can be solved using modular arithmetic. If the number of peanuts Dumbo is given is N, $N \equiv 3 \pmod 7$ and $N \equiv 9 \pmod{11}$. If these statements are rewritten in equation form, $N = 7a + 3 = 11b + 9$ for some integers a and b. This means $4b + 2 \equiv 3 \pmod 7$ which simplifies to $b \equiv 2 \pmod 7$. That means $b = 7c + 2$ for some integer c and $N = 77c + 31$. Since $N < 300$ that means $77c + 31 < 300$ and $c < 3\frac{3}{77}$. Therefore, to maximize N, $c = 3$ and $N = \boxed{262}$.

33. Consider the angular velocity of the two ants. The first ant travels 3 laps an hour, equivalent to $\frac{1}{20}$ of a lap per minute. Similarly, the second ant travels $\frac{1}{12}$ of a lap per minute. Since a lap represents 360 degrees around the clock, the angle between the ants grows at a rate of $(\frac{1}{12} - \frac{1}{20})360° = 12°$ a minute. The

starting angle between the two ants is of course $0°$. The next time the ants are next to each other, the angle between them will have grown to $360°$. The angle grows at $12°$ a minute, so it will be 30 minutes before the ants are next to one another again. In 30 minutes, the first ant will have traveled $1\frac{1}{2}$ laps, and the second will have traveled $2\frac{1}{2}$ laps; they will both be on the $\boxed{6}$.

34. Say that Autumn sleeps for x hours. Then she has $18 - x$ hours during the day that she can work on her problem set. The amount of work she can do, denoted by y, is $18 - x$ multiplied by her speed. Her speed is $1 - 0.10(9 - x) = 0.10 + 0.10x$ since she loses 10% of her normal speed for every hour less than 9 that she sleeps and gains 10% for every hour more than 9 that she sleeps. Then we know $y = (18 - x)(0.10 + 0.10x)$. The roots are $x = -1, 18$, so the vertex (maximum) must be at $x = 8.5$ hours, so the answer is $\boxed{85}$.

35. First, we will pretend that the balls are indistinguishable, and take this into account later. Since each friend must receive at least one ball, give each of them one initial ball, leaving 5 balls to hand out. Now, through simple trial-and-error, we can find that the only way to distribute 5 balls amongst 3 people so that they all get different amounts is (0,1,4) or (0,2,3) (each in some order). So, with the initial 8 balls, these can be distributed as (1,2,5), (1,3,4), or some permutation of those.

    Now, we will consider distinguishable balls. If we distribute the balls as (1,2,5), then we have $3! = 6$ ways to choose which friend gets which number of balls. Then, we have 8 choices for which ball to give to the friend who gets 1 ball, and $\binom{7}{2} = 21$ ways to choose which two remaining balls to give the friend who gets 2 balls. The third friend gets the remaining balls. So, in this case, there are $6 \cdot 8 \cdot 21$ ways.

    In the second case of (1,3,4), there are 6 ways to choose which friend gets which number, 8 ways to give a ball to the first friend, $\binom{7}{3} = 35$ ways to give 3 remaining balls to the second friend, and the third friend gets the remaining. So, that's $6 \cdot 8 \cdot 35$ ways here.

    In total, we have $\boxed{2688}$ distributions.

36. Because $8, 9$, and 10 are consecutive, $10 \equiv 1 \pmod 9$ and $8 \equiv -1 \pmod 9$. Thus, base-10 $abcd = 1000a + 100b + 10c + d \equiv a + b + c + d \equiv 0 \pmod 9$, and base-8 $abcd = 512a + 64b + 8c + d \equiv -a + b - c + d = (b + d) - (a + c) \pmod 9$. For base-8 $abcd \equiv 0 \pmod 9$, we need $b + d \equiv a + c \equiv 0 \pmod 9$.

    Since $1 \le a, b, c, d \le 7$, $a + c = b + d = 9$. Hence, there are 6 possible values for a and b (between 2 and 7), To make the sums 9, $c = 9 - a$ and $d = 9 - b$, so there are no additional choices to make. Thus, there are $6 \cdot 6 = 36$ ways to choose a, b, c, d, so there are $\boxed{36}$ sequences $abcd$.

37. We have that $2^a \equiv 9 \implies 2^{ab} \equiv 9^b \equiv 3^{2b} \equiv 4^2 \pmod{71}$, and $3^b \equiv 4 \implies 3^{ab} \equiv 4^a \equiv 2^{2a} \equiv 9^2 \pmod{71}$. So, $6^{ab} \equiv 2^{ab} \cdot 3^{ab} \equiv 16 \cdot 81 \equiv 16 \cdot 10 \equiv \boxed{18} \pmod{71}$.

38. We add 4 to both sides of each equation and use Simon's Favorite Factoring Trick to rewrite them as

$$(a + 2)(b + 2) = 216 = 2^3 \cdot 3^3$$
$$(b + 2)(c + 2) = 90 = 2 \cdot 3^2 \cdot 5$$
$$(c + 2)(d + 2) = 200 = 2^3 \cdot 5^2.$$

    Since there are no powers of 3 in the last equation, we see that the 3^2 in the second equation must be a divisor of $b + 2$. We also see that since 216 contains no powers of 5, then $b + 2$ contains no powers of 5. Therefore, from the second equation, $b + 2 = 9$ or $b + 2 = 9 \cdot 2 = 18$.

    The first case gives us $c + 2 = 10, a + 2 = 24$, and $d + 2 = 20$. Therefore, $a + d = 40$. The second case gives us $c + 2 = 5, a + 2 = 12$, and $d + 2 = 40$. Therefore, $a + d = 48$. Hence, the answer is $40 + 48 = \boxed{88}$.

39. We will substitute $m = n + 2$ into the equation. Thus we have $\frac{(m-2)^3+50}{m} = m^2 - 6m + 12 - \frac{42}{m}$. Since this value must be an integer, we have that $\frac{42}{m}$ must be an integer. The largest factor of 42 is 42, so $m = 42$ yields $n = \boxed{40}$.

40. Note that since MN is a midline of $\triangle ABC$, $AC \parallel MN$ and $AC = 5 \cdot 2 = 10$. Then, since $\angle MND$ is a right angle by the Pythagorean Theorem, lines AC and ND are also perpendicular. Thus, the area of $ANCD$ can be computed as $10 \cdot 12/2 = 60$. This area can be split up into $[ADC] + [ANC] = 27 + [ANC]$, which implies that $[ANC] = 33$. Finally, since N is the midpoint of BC, the total area can be computed as $[ABCD] = [ADC] + [ABC] = 27 + 2 \cdot 33 = \boxed{93}$.