Joe Holbrook Memorial Math Competition

8th Grade

October 17, 2021

General Rules

- You will have **75 minutes** to solve **40 questions**. Your score is the number of correct answers.
- Only answers recorded on the answer sheet will be graded.
- This is an individual test. Anyone caught communicating with another student will be removed from the exam and their score will be disqualified.
- Scores will be posted on the website. Please do not forget your ID number, as that will be the sole means of identification for the scores.
- You may use the following aids:
  - Pencil or other writing utensil
  - Eraser
  - Blank scrap paper
- You may not use the following aids:
  - The Internet
  - Books or other written sources
  - Other people
  - Calculator or other computing device
  - Compass
  - Protractor
  - Ruler or straightedge

Other Notes

- Please input your answers into the Google form provided by your proctor.
- All answers are integers. Make sure you do not make any typing mistakes, as you will not be given credit if you do so.
- You do not need to write units in your answers.
- Ties will be broken by the number of correct responses to questions 31 through 40. Further ties will be broken by the number of correct responses in the last five questions.

1. Detective Spencer is trying to catch a criminal before they get away! Using his skills, he has deduced that they are definitely either at the mall, the airport, or the subway station. He calculates that there is a 24% chance they are at the mall, and a 52% chance they are at the airport. What is the percent chance that they are at the subway station?

2. Jan buys 8 books for \$6 each. Then, she decides to sell those books for \$9 each. How many books does she have to sell so that she makes more money than she spent?

3. Galadriel is reading a book. On Monday she reads $\frac{1}{4}$ of the book, on Tuesday she reads $\frac{1}{3}$ of the book, and on Wednesday she reads the last 55 pages of the book. How many pages are there in the book?

4. Nikhil loves potatoes and visits the Potato Cafe. At the Potato Cafe, he gets his potato: baked, fried, or mashed. In addition to these options, he can also add one topping: broccoli, cheese, sour cream, or butter. He could also choose to get no topping at all. How many possible ways can Nikhil customize his potato?

5. In a class of fifty 11th and 12th graders, after 40% of the 12th graders leave, there are 42 students left. How many juniors are there?

6. How many ways can we pick two numbers from $\{1, 2, 3, 4, 5\}$ so that their sum is odd? (Order does not matter, so picking 1 then 2 is the same as 2 then 1.)

7. A tortoise and a hare decide to race. The tortoise walks at a steady pace of x miles per hour while the hare decides to run at 25 miles per hour for the first half of the race distance and the 5 mph for the second half. If the race is going to be 30 miles long, what is the minimum integer value of x so that the tortoise beats the hare?

8. The product of the digits of a ten-digit number is 105. What is the sum of these digits?

9. Alice is very good at math. Alice solves 10 problems every hour, while Autumn only solves 4 problems every hour. If they solve problems at a constant rate throughout each hour, how many minutes will it take for them to solve a total of 35 problems?

10. Primes p_1, p_2, p_3 satisfy $p_1 + p_2 = 43$ and $p_1 + p_3 = 45$. Find the sum $p_2 + p_3$.

11. Nikhil wants to make a viral video by rolling two fair icosahedral dice (20-sided dice) and having both land on 20. Eventually, he succeeds! To prove the dice aren't rigged, he rolls them again, hoping to get no 20s. Let the probability he doesn't roll a 20 on either die be p. If $p = \dfrac{n}{m}$ with n, m in lowest terms, then what is $n + m$?

12. Alicia is burning a cylinder-shaped candle with radius of 5 cm and height of 10 cm at a rate of $\dfrac{1}{2}\text{cm}^3$ per minute. How many minutes, to the closest integer, will it take for the candle to have height of 6 cm left? (Note: Assume $\pi = 3.14$.)

13. A collection of ten positive integers has a unique mode of 5 and a mean of 5.4. Eight of the integers are 1, 3, 4, 5, 5, 7, 9, 13. What is the product of the two remaining integers?

14. There are four different senior English teachers at BCA, and each student has an equal probability of being assigned to each teacher. The probability that Autumn, David, and Erez are all assigned to the same teacher is $\dfrac{a}{b}$, where a and b are positive relatively prime integers. What is $a + b$?

15. How many two digit numbers satisfy the property that the sum of their digits is a multiple of 7?

16. Alicia and Yul are taking a walk together. Alicia is a law-abiding citizen: from point A, she walks directly across a 15-foot street, then turns 90° to the left and walks on the sidewalk for 20 more feet to reach point B. However, Yul insists on walking in the middle of the street, so she walks directly from point A to point B. Assuming they both walk in straight lines, how many more feet does Alicia have to walk?

17. How many nonzero terms are in the expansion (the fully-distributed and simplified form) of

$$(x + 1)(x + 2)(x + 3) \dots (x + 9)(x + 10)?$$

18. If x is an integer such that $x = \sqrt{2x + 8}$, what is the sum of all possible values of x?

19. Autumn likes bothering David. Every time she pokes David, the chance that he throws his stuffed turtle at her increases additively by $\frac{1}{10}$, beginning at $\frac{1}{10}$ the first time she pokes him. Once he throws his stuffed turtle at her, she stops poking him. The probability David throws his stuffed turtle at her the third time she pokes him is $\frac{a}{b}$, where a and b are positive relatively prime integers. What is $a+b$?

20. Allen draws a square. He then inscribes a circle within the square, inscribes another square in that circle, and finally inscribes a circle within the smaller square. The ratio of the area of the smaller circle to the area of the original square can be written in the form $\frac{a\pi}{b}$, where $\gcd(a,b)=1$. Find $a+b$.

21. The common ratio in geometric sequence a_n is $\dfrac{1}{2}$. If a_m is 1 and the sum of the terms from the first term to the mth term is 4095, what is the value of m?

22. Erez loves cookies-and-cream ice cream. At his ice cream parlor, you can choose from the flavors vanilla, chocolate, strawberry, and cookies-and-cream, but he requires you to buy at least one scoop of cookies-and-cream. You can buy multiple scoops of the same flavor, and different arrangements of the same scoops are considered the same. If you buy 3 scoops of ice cream on a cone, how many possible orders can you get?

23. Farmer John ties his two cows, Bessie and Elsie, to a small shed. The base of the shed is rectangle $ABCD$, where $AB=4$ yards and $BC=2$ yards. John ties Bessie to corner A with a 3 yard rope and ties Elsie to corner B with a 1 yard rope. If the combined area of grass that Bessie and Elsie can graze on in square yards is $\dfrac{a\pi}{b}$ in lowest terms, then what is $a+b$?

24. A three-digit positive integer n has six distinct factors, one of which is 49. How many possible values are there for n?

25. Michael is bored at home. He flips a coin twice, and he gets heads (H) then tails (T). He decides he will continue flipping the coin as long as it follows the pattern HTHTHTHTHT... What is the expected number of times he will flip the coin, including the first two flips?

26. Marshmallow kicks Olaf off of a 20 foot tall cliff. The equation for Olaf's height h at time t seconds after Olaf is kicked is $h(t)=at^2+bt+c$, for some values a,b,c where h is in feet and t is in seconds. It takes 3 seconds for Olaf to reach his maximum height and 17 more seconds for Olaf to land at the base of the cliff. Find $\dfrac{3ac}{b}$.

27. A math contest has 40 questions to be completed in 75 minutes. 14 problems are considered simple, 13 are medium, and 13 are hard. Each simple problem is 2 points, each medium problem is 3 points, and each hard problem is 5 points. It takes David 2 minutes to complete a simple problem, 5 minutes to complete a medium one, and 8 minutes to complete a hard one. What is the maximum score David can get?

28. There are 2 cards, one of which is purple on both sides, and the other of which is purple on one side and green on the other side. Autumn randomly picks a card and looks at one side. If the side she sees is purple, the probability that the other side of that card is also purple is $\dfrac{a}{b}$, where a and b are positive relatively prime integers. What is $a+b$?

29. There is a geometric sequence with positive first term a_1 and ratio r. If $a_2=\text{lcm}(a_1,r)^2$, and a_{17} is 17^{34}, what is $a+r$?

30. Dory is counting the number of fish in a coral reef, but she forgets that the digit 3 exists and skips it while counting. For example, she will count 629, then 640, skipping all the numbers in between because they include a 3. If she counts 456 fish in the reef, how many fish are actually there?

31. Point P is in a circle of radius 3 and is 2 units away from the circle's center O. The set of midpoints of all the chords of the circle that pass through P has area M. Find $\lfloor M\rfloor$.

32. Emily moves to Octoville, where everyone uses base-8. She lives in a house numbered $abcd$, where a,b,c,d are not necessarily distinct integers and $1\le a,b,c,d\le 7$. She observes that the base-10 number $abcd$ is divisible by 9. How many sequences $abcd$ are there such that the base-8 number $abcd$ is also divisible by 9?

33. There exist integers a and b such that 2^a and 3^b have remainders of 9 and 4, respectively, when divided by 71. What is the remainder when 6^{ab} is divided by 71?

34. Square ABCD lies on the coordinate plane so that $\overline{AB}$ is on the x-axis and C and D are points on the parabola $y = -x^2 + 15$. Find the sum of all possible areas of $ABCD$.

35. Let $f(x) = \dfrac{x^2 + 2x + 3}{x^2 + 4x + 6}$. If $f(-7)f(-6)\cdots f(6)f(7)$ is $\dfrac{a}{b}$ in simplest form, what is $a + b$?

36. What is the largest integer n such that $n^3 + 50$ is a multiple of $n + 2$?

37. There are two distinguishable boxes A, B and 20 red balls, 20 yellow balls, and 20 green balls. In how many ways can you place the 60 balls in 2 boxes so that each box has 30 balls? (Note: Same colored balls are indistinguishable.)

38. Let $ABCD$ be a regular tetrahedron with side length $AB = \sqrt{2}$. If P is in the interior of $\triangle ABC$, $AP = BP$, and $CP = 1$. If $d = DP$ and $3d^2 = n - \sqrt{m}$, what is $n + m$?

39. Let the plane be divided into some amount of regions by the lines $y = 0, x = 0, y = -x + 3, y = \dfrac{x}{2} - 7$. Define a jump as hopping over a line from a region to an adjacent region. We start at a random region and color it blue. We then jump to across a random edge and color the new region red. We jump again, color the region blue, and continue alternating until all regions have been labeled. Note that if we revisit a colored region, we can recolor it with our new color. Let the expected number of regions labeled blue at the end be in the form $\dfrac{a}{b}$, where $\gcd(a, b) = 1$. What is $a + b$?

40. Let $ABCD$ be a convex quadrilateral with points M and N being the midpoints of AB and BC respectively. If $MN = 5$, $DN = 12$, $DM = 13$, and the area of triangle ADC is 27, compute the area of $ABCD$.