Joe Holbrook Memorial Math Competition

8th Grade Solutions

October 17, 2021

1. Since the criminal is definitely at one of the mall, the airport, or the subway station, there is a 100%chance they are at one of them. Then, if we let x be the percent chance that they are at the subway station, we have $24\% + x\% + 52\% = 100\%$. So, $x + 76 = 100 \implies \boxed{x = 24\%}$.

2. She spent \$48. She needs to sell $\boxed{6}$ books so that she sells \$54 worth of books.

3. "Suppose the book has n pages. Because she finishes the book, we know that

$$\frac{1}{4}n + \frac{1}{3}n + 55 = n.$$

Simplifying, we see

$$\frac{5}{12}n = 55$$

so $n = 55 \cdot \frac{12}{5} = \boxed{132}$ pages.

4. 3 ways of getting his potato and 5 ways of choosing a topping (or none at all) yields $3 \cdot 5 = \boxed{15}$ ways.

5. Since 50-42=8 students leave, 40% of the 12th graders is 8 people, meaning that there are $\frac{1}{0.4} \cdot 8 = 20$ 12th graders. Then, there are $50 - 20 = \boxed{30}$ 11th graders.

6. If the sum is odd, one number must be even and the other must be odd. There are 3 odd numbers to choose from and 2 even, giving $2 \cdot 3 = \boxed{6}$.

7. The total time it takes for the hare to complete the first half of the race is $\frac{15}{25} = \frac{3}{5}$ hours. The time it took the hare to complete the second half of the race is $\frac{15}{5} = 3$ hours. In total that is $\frac{3}{5} + 3 = \frac{18}{5}$ hours. If the tortoise runs the race at a constant speed, he should run at $\frac{30}{\frac{18}{5}} = \frac{30 \cdot 5}{18} \leq \boxed{9}$ miles per hour.

8. The prime factorization of 105 gives $3 \cdot 5 \cdot 7$. The product of any pair of these gives a result greater than 9, which would not be a digit. Therefore 3 of the digits are 3, 5, and 7, and the rest are 1s. The sum is $3 + 5 + 7 + 7 \cdot 1 = \boxed{22}$.

9. Since Alice solves 10 problems per hour and Autumn solves 4 problems per hour, together they solve a total of 14 problems per hour. Then solving 35 problems will take $\frac{35}{14} = 2.5$ hours $= \boxed{150}$ minutes.

10. Since the sum of p_1 and p_2 is odd, one of them must be even and the other must be odd. The only even prime is 2. Since the sum of p_1 and p_3 is also odd, we must have $p_1 = 2$. Thus $p_2 = 41$ and $p_3 = 43$, so the answer is $\boxed{84}$.

11. The probability of rolling no 20s is the same whether or not Nikhil rolls a pair of 20s before the proof roll. The probability of getting a not-20 on one die is $\frac{19}{20}$, so the probability of getting a not-20 on both dice is $\frac{19}{20} \cdot \frac{19}{20} = \frac{361}{400}$, so the answer is $\boxed{761}$.

12. The candle must have burned the top $4cm$ of the candle. The volume of this would be calculated as follows: $5^2 * \pi * 4 = 100 * \pi \approx 314$. Since the candle burns $\frac{1}{2}\text{cm}^3$ per minute, it will take $\boxed{628 \text{ minutes}}$.

13. Since the mean of the ten numbers is 5.4, we know that the sum of the numbers is $5.4 \cdot 10 = 54$. The sum of the given eight integers is $1+3+4+5+5+7+9+13 = 47$. This means that the sum of the remaining two numbers is $54 - 47 = 7$. If the two integers were 1 and 6, the mode would not be unique. Similarly if the two integers were 3 and 4, the mode would not be unique. Thus the only possibility is if the two integers are 2 and 5. This gives us a product of $\boxed{10}$.

14. There are $4 \cdot 4 \cdot 4 = 64$ possible ways to assign a teacher to Autumn, David, and Erez. There are four different teachers that the three students could all be assigned to, so 4 of the 64 assignments allow the students to have the same teacher. This yields a probability of $\frac{4}{64} = \frac{1}{16}$, which gives us an answer of $1 + 16 = \boxed{17}$.

15. The sum of the digits must be 7 or 14 (it cannot be 21, as the maximum possible sum of the digits in a two digit number is 18). If the sum is 7, the number could be $16, 25, 34, 43, 52, 61$, or 70, and if the sum is 14, the number could be $59, 68, 77, 86$, or 95, which is $\boxed{12}$ numbers in total.

16. Notice that the paths of Alicia and Yul form a right triangle with legs 15 feet and 20 feet, where Alicia walks the legs and Yul walks the hypotenuse. We can notice that this is a 3-4-5 triangle to find that the hypotenuse is 25 feet. Thus, Alicia must walk $(15 + 20) - 25 = \boxed{10}$ more feet than Yul.

17. Every term in the expansion will be positive, so all powers of x between 0 and 10 will be present. Thus, there will be a constant term, an x term, an x^2 term, and so on up to the x^{10} term, for a total of $\boxed{11}$ terms.

18. Square both sides of the equation to get $x^2 = 2x+8$, or $x^2 - 2x - 8 = 0$. There are many ways to solve this equation, but one way is to use the quadratic formula: $x = \frac{2 \pm \sqrt{36}}{2} = 4$ or -2. In the original equation, we have that x is the square root of something, so it must be positive, and our answer is $\boxed{4}$.

19. The probability that David does not throw his stuffed turtle at Autumn the first time she pokes him is $1 - \frac{1}{10} = \frac{9}{10}$, and the probability that he also does not throw his stuffed turtle at her the second time she pokes him is $1 - \frac{2}{10} = \frac{8}{10}$. Finally, the probability that he does throw his stuffed turtle at her the third time is $\frac{3}{10}$. This yields a probability of $\frac{9}{10} \cdot \frac{8}{10} \cdot \frac{3}{10} = \frac{27}{125}$ which gives us an answer of $27 + 125 = \boxed{152}$.

20. Let the side length of the original square be 1, so the area of the smaller circle will be the ratio. Notice the diameter of the larger circle is 1, so the radius is $\frac{1}{2}$. Then the radius of the smaller circle can be calculated using an isosceles right triangle with a radius of the large circle as the hypotenuse and a radius of the small circle as a leg. We get $r = \frac{1}{2\sqrt{2}}$, which results in an area of $\frac{\pi}{8} \implies a + b = 1 + 8 = \boxed{9}$.

21. Constructing the series from a_m towards a_1, we get a sequence that looks like this: $1, 2, 4, 8, \cdots$. We want this to add up to 4095. Note that the sum of powers of 2 from 2^0 to 2^n is equal to $2^{(n+1)} - 1$. We know that $2^{12} = 4096$. Thus $4095 = 2^0 + 2^1 + \cdots + 2^{11}$ and the first term would be $a_{(m-11)}$ which means that the value of m is $\boxed{12}$.

22. We split this into cases.
Case 1: You buy exactly 1 scoop of cookies-and-cream. Then for the last 2 scoops: you can have 2 different flavors, which is $\binom{3}{2} = 3$ choices, or the same flavor, which is $\binom{3}{1} = 3$ choices. (Or you can simply list out all the combinations!) This is 6 choices in total.
Case 2: You buy exactly 2 scoops of cookies-and-cream. Then there are 3 choices for the 3rd scoop.
Case 3: You buy exactly 3 scoops of cookies-and-cream. There is only 1 choice for this.
Adding all 3 cases, we get a total of $6 + 3 + 1 = \boxed{10}$ orders in total.

23. Bessie can graze on $\frac{3}{4}$ of a 3-radius circle and $\frac{1}{4}$ of a 1-radius circle, while Elsie can graze on $\frac{3}{4}$ of a 1-radius circle. Thus, the area is $\frac{3}{4}\pi \cdot 3^2 + \frac{4}{4}\pi \cdot 1^2 = \frac{31\pi}{4}$ and the answer is $31 + 4 = \boxed{35}$.

24. We know that $n = 7^2 m$ where m is a positive integer. For a number to consist of six factors, it must be of the form a^5 or $a^2 b$ for distinct primes a and b. It is obvious that the first option will cause n to exceed the three-digit limit, so n must follow the second format. Thus, m must be a prime not equal to 7 where $49m$ is a three-digit number. The possible values for m are 3, 5, 11, 13, 17, and 19, for a total of $\boxed{6}$ values.

25. We can ignore the first two flips and add 2 to our final answer to compensate. Because the probabilities for H and T are equal, we can effectively invert every other coin flip, and the problem becomes "if a person keeps flipping a coin as long as they get heads, what is the expected number of flips they will make?". The expected value for that problem is $E = 1 + \frac{1}{2}E + \frac{1}{2}0 \Rightarrow E = 2$, and we must add two to get our final answer of $\boxed{4}$.

26. Based on the problem statement, $h(t)$ contains the points $(0, 20)$ because Olaf is 20 feet high at time 0 and $(20, 0)$ because Olaf hits 0 feet after $3 + 17 = 20$ seconds. Also note that the t-coordinate of the vertex of the $h(t)$ is 3. That means the other t-intercept is $(-14, 0)$. So $h(t) = m(t - 20)(t + 14)$ for some value m. By plugging in the point (0,20) we can find that $m = -\frac{1}{14}$. Therefore, $h(t) = -\frac{1}{14}(t - 20)(t + 14) = -\frac{1}{14}t^2 + \frac{3}{7}x + 20$, so $a = -\frac{1}{14}$, $b = \frac{3}{7}$, and $c = 20$. Therefore, $\frac{3ac}{b} = \boxed{-10}$.

27. To maximize his score in a given time, David should spend his time as efficiently as possible. While doing simple problems, he earns 2 points every 2 minutes, or 1 point per minute. While doing medium problems, he earns 3 points every 5 minutes, or $\frac{3}{5}$ of a point per minute. Finally, while doing hard problems, he earns $\frac{5}{8}$ of a point per minute.

    Accordingly, David should first do simple problems. There are 14 simple problems, and David has the time to solve all of them, so David will spend 28 minutes earning 28 points. After he is done, there are $75 - 28 = 47$ minutes remaining.

    At first glance, the ideal set-up might seem to be doing as many hard problems as possible. In this set-up, David does 5 hard problems in the 47 minutes remaining. With the 7 minutes he has left, he finishes one last medium problem with 2 minutes to go. In total, David would earn 56 points. However, in the last two minutes, David earns no points at all; this suggests a better solution is possible.
    David can avoid wasting time by doing 4 hard problems instead of 5. This leaves 15 minutes, which David can use to do 3 medium problems. Under this set-up, David earns $28 + 20 + 9 = \boxed{57}$ points, narrowly edging out the first solution. Doing any fewer hard problems will lead to a lower score, so this is the maximum score David can get.

28. There is a $\frac{1}{2} \cdot \frac{2}{2} = \frac{1}{2}$ chance that Autumn chose the card with both purple sides and sees a purple side and a $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ chance that she chose the card with one purple side and sees the purple. The probability that she chose the card with both purple sides is thus $\frac{\frac{1}{2}}{\frac{1}{2} + \frac{1}{4}} = \frac{2}{3}$, so our answer is $2 + 3 = \boxed{5}$.

29. Since $a_2 = a_1 r$, and using the fact that $a \cdot b = \text{lcm}(a, b) \gcd(a, b)$, we have that

$$a_1 r = \text{lcm}(a_1, r) \cdot \gcd(a_1, r) = \text{lcm}(a_1, r)^2$$

    which gives that $\gcd(a_1, r) = \text{lcm}(a_1, r)$. This implies that $a = r$ so $a_{17} = ar^{16} = 17^{34} \implies a = r = 289$, so our answer is $\boxed{578}$.

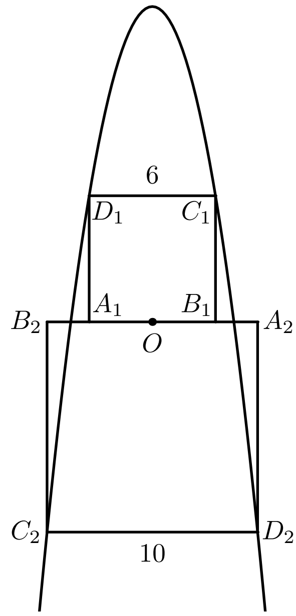
30. Dory is basically counting in a form of base 9 where when she counts $1, 2, 4, 5, 6, 7, 8, 9$ she is counting the base 9 equivalent of $1, 2, 3, 4, 5, 6, 7, 8$. Therefore, the number 456 in Dory's counting method is like the number 345_9. Now, to get the real number of fish just convert 345_9 to base 10. $345_9 = 9^2 \times 3 + 9^1 \times 4 + 9^0 \times 5 = 284_{10}$. Therefore, there are actually $\boxed{284}$ fish in the coral reef.

31. The line from the centers to the midpoints of these chords form right angles, so this region is just a circle whose diameter is OP. Its radius is 1, so the area is π and $\lfloor \pi \rfloor = \boxed{3}$.

32. Because $8, 9$, and 10 are consecutive, $10 \equiv 1 \pmod 9$ and $8 \equiv -1 \pmod 9$. Thus, base-10 $abcd = 1000a + 100b + 10c + d \equiv a + b + c + d \equiv 0 \pmod 9$, and base-8 $abcd = 512a + 64b + 8c + d \equiv -a + b - c + d = (b + d) - (a + c) \pmod 9$. For base-8 $abcd \equiv 0 \pmod 9$, we need $b + d \equiv a + c \equiv 0 \pmod 9$.

    Since $1 \leq a, b, c, d \leq 7$, $a + c = b + d = 9$. Hence, there are 6 possible values for a and b (between 2 and 7), To make the sums 9, $c = 9 - a$ and $d = 9 - b$, so there are no additional choices to make. Thus, there are $6 \cdot 6 = 36$ ways to choose a, b, c, d, so there are $\boxed{36}$ sequences $abcd$.

33. We have that $2^a \equiv 9 \implies 2^{ab} \equiv 9^b \equiv 3^{2b} \equiv 4^2 \pmod{71}$, and $3^b \equiv 4 \implies 3^{ab} \equiv 4^a \equiv 2^{2a} \equiv 9^2 \pmod{71}$. So, $6^{ab} \equiv 2^{ab} \cdot 3^{ab} \equiv 16 \cdot 81 \equiv 16 \cdot 10 \equiv \boxed{18} \pmod{71}$.



34. Label the origin as O. Suppose B has coordinates $(b, 0)$. Because $\overline{BC}$ is parallel to the y-axis and C is on the parabola, C must have coordinates $(b, 15 - b^2)$. $\dfrac{BC}{BO} = 2$ because ABCD is a square, so $\dfrac{15 - b^2}{b} = 2$. We rearrange this to get the quadratic $b^2 + 2b - 15 = 0$, which has solutions $b = 3, -5$. These cases represent the two possible configurations with the square inside and outside the parabola.
If $b = 3$, then $c = 15 - b^2 = 6$. Thus, $BC = 6$ and $ABCD = 6^2 = 36$. If $b = -5$, then $c = 15 - b^2 = -10$. Thus, $BC = |-10| = 10$ and $ABCD = 100$. Therefore, the sum of the two possible areas is $36 + 100 = \boxed{136}$.

35. We can factor the expression as $\dfrac{(x+1)^2 + 2}{(x+2)^2 + 2}$, which makes it clear that this is a telescoping product. All intermediate terms thus cancel, and the product is $\dfrac{(-6)^2 + 2}{9^2 + 2} = \dfrac{38}{83}$, so the answer is $38 + 83 = \boxed{121}$.

36. We will substitute $m = n + 2$ into the equation. Thus we have $\dfrac{(m-2)^3 + 50}{m} = m^2 - 6m + 12 - \dfrac{42}{m}$. Since this value must be an integer, we have that $\dfrac{42}{m}$ must be an integer. The largest factor of 42 is 42, so $m = 42$ yields $n = \boxed{40}$.

37. Let a be the number of red balls going into box A, b be the number of yellow balls going into box A, and c be the number of green balls going into box A. Then we can calculate the possible triples (a, b, c) that satisfies $a + b + c = 30$ and subtract cases where $a > 20, b > 20$ or $c > 20$. Using stars and bars, we get that there are $\dbinom{32}{2} = 496$ ways. Now let's calculate the case where $a > 20$. If $a = 21$, then $b + c = 9$ and there are 10 possibilities for b and c. If $a = 22$, then $b + c = 8$ and there are 9 possibilities. We use the same logic until we get to $a = 30$ and $b + c = 0$ which would give 1 possibility. Thus, there are $10 + 9 + \cdots + 2 + 1 = 55$ possibilities. The same logic applies to when $b > 20$ and $c > 20$ so we subtract $55 * 3 = 165$ from 496 to get $\boxed{331}$.

38. This problem was built to lend itself nicely to coordinate geometry. The tetrahedron of side length $\sqrt{2}$ fits snugly inside the unit cube, and to make our lives easy we can choose $D = (0,0,0)$, $A = (0,1,1)$, $B = (1,0,1)$, and $C = (1,1,0)$. This means that the distance DP is just the distance of P from the origin, with $P = (x, y, z)$ satisfying the following conditions:

- P is in the interior of $\triangle ABC$, which itself is part of the plane $x + y + z = 2$.
- P is a distance of 1 from C, so $(x-1)^2 + (y-1)^2 + z^2 = 1$.
- $AP = BP$, which means the system is symmetric across $x = y$, and thus we have that P lies in the plane $x = y$.

This gives us a system of three equations that are easy to solve through substitutions:

$$\begin{aligned} x &= y \\ x + y + z = 2 &\Rightarrow 2x + z = 2 \\ &\Rightarrow z/2 = 1 - x \\ (x-1)^2 + (y-1)^2 + z^2 = 1 &\Rightarrow (z/2)^2 + (z/2)^2 + z^2 = 1 \\ &\Rightarrow 6z^2 = 4 \\ &\Rightarrow z = \sqrt{2/3} \\ z/2 = 1 - x &\Rightarrow x = 1 - \sqrt{1/6} \end{aligned}$$

Thus we have:

$$3d^2 = 3\left(2\left(1 - \sqrt{\frac{1}{6}}\right)^2 + \frac{2}{3}\right) = 9 - \sqrt{24}$$

and the answer is $9 + 24 = \boxed{33}$.

Solution 2: Let O be the center of $\triangle ABC$. Since $OA = OB$, and $CA = CB$, we have that O, C, and P all lie on the perpendicular bisector of AB. Furthermore, by dropping an altitude from O to AC, we form a 30-60-90 triangle, and can find that $OC = \frac{\sqrt{6}}{3}$. So, $OP = 1 - \frac{\sqrt{6}}{3} = \frac{3-\sqrt{6}}{3}$.

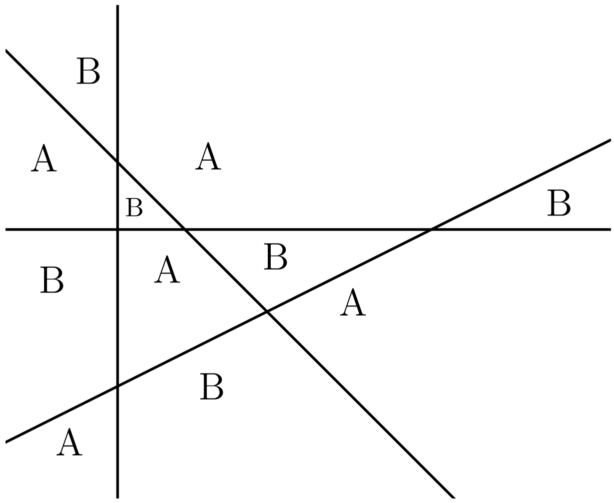
Now, consider triangle AOD. Since D is equidistant from A, B, C, and O is the center of equilateral triangle ABC, it should be intuitive that DO is perpendicular to the plane of $\triangle ABC$, and in particular, $DO \perp OA$. Then, since $OA = OC = \frac{\sqrt{6}}{3}$ and $AD = \sqrt{2}$, we can compute, via Pythagorean theorem, that $DO^2 = \frac{4}{3}$.

Finally, since DO is perpendicular to the plane of $\triangle ABC$, $DO \perp OP$, so we can compute DP^2 with Pythagorean theorem as:

$$DP^2 = DO^2 + OP^2 = \frac{4}{3} + \frac{15 - 6\sqrt{6}}{9} = \frac{9 - 2\sqrt{6}}{3}.$$

So, $3d^2 = 9 - \sqrt{24}$, and the answer is $\boxed{33}$

39. If we play around with the lines, we can see that whenever a region is revisited, it will always have the same color; this is because all cycles have an even number of crossings. This means that there are two classes of regions: the A regions that will be colored the same as the central region (the region including $(1, -1)$), and B regions that will be colored opposite the central region. There are 5 regions in the A class and 6 regions in the B class. Thus, there's a $\frac{5}{11}$ chance we start in the A class, and then 5 regions will be blue. Alternatively, there's a $\frac{6}{11}$ chance we start in the B class, and then 6 regions will be blue. Thus, the expected number of blue regions is $\frac{5}{11} \cdot 5 + \frac{6}{11} \cdot 6 = \frac{61}{11}$ so our answer is $61 + 11 = \boxed{72}$.



40. Note that since MN is a midline of $\triangle ABC$, $AC \parallel MN$ and $AC = 5 \cdot 2 = 10$. Then, since $\angle MND$ is a right angle by the Pythagorean Theorem, lines AC and ND are also perpendicular. Thus, the area of $ANCD$ can be computed as $10 \cdot 12/2 = 60$. This area can be split up into $[ADC] + [ANC] = 27 + [ANC]$, which implies that $[ANC] = 33$. Finally, since N is the midpoint of BC, the total area can be computed as $[ABCD] = [ADC] + [ABC] = 27 + 2 \cdot 33 = \boxed{93}$.