Joe Holbrook Memorial Math Competition

5th Grade Solutions

March 20, 2022

1. If Train A leaves from the station every 30 minutes and Train B leaves from the station every 18 minutes, this means that Train A and B leave the station together every LCM(30,18)= 90 minutes. Since both of the trains left the station at 1:00 PM, this means that the next time that the trains will leave at the same time is 2:30 PM. Tommy and Timmy arrived at the station at 1:30, which means they will have to wait $\boxed{60}$ minutes for the trains to leave the station at the same time again, at 2:30 PM.

2. One option is to just try all numbers starting with 2, until we find 19. Alternatively, notice that if x is divisible by 2 or 3, then $x+6$ will be as well, so the smallest prime factors of x and $x+6$ will be the same. So, we can instead restrict our search to x not divisible by 2 or 3 (equivalently, 1 or 5 mod 6). Then, we have to check $x = 1, 5, 7, 11, 13, 17$, until we find $x = \boxed{19}$, since the smallest prime factor of 19 is 19, while the smallest prime factor of 25 is 5.

3. The fraction of books that are on miscellaneous topics is $1 - \frac{1}{16} - \frac{2}{7} - \frac{1}{2} - \frac{1}{8} = \frac{3}{112}$. That means there are $224 \times \frac{3}{112} = 6$ books on miscellaneous topics. There are $6! = \boxed{720}$ ways to arrange the books.

4. There are 7 consonants and 3 vowels. Because the first and last letters are consonants, the 2nd and 4th letters must be vowels (as there must be at least one vowel between two consonants.) The middle letter can be either a vowel or a consonant for the word to still work, so there are $7 \cdot 7 \cdot 3 \cdot 3 \cdot 10 = \boxed{4410}$ options.

5. Denote Moana's speed as m and the wind's speed as w. Since distance is equal to speed times time. $21 = (m + w) \times 3$ and $21 = (m - w) \times 4$. If the system of equations is solved, it can be found that $w = \frac{7}{8}$. So, the answer is $7 + 8 = \boxed{15}$.

6. Firstly, we must find the areas of the two different colors. One color will be made of the portions of the rings with 1 ,3 ,5 , and 7 inch radii. The respective areas are π, $9\pi - 4\pi$, $25\pi - 16\pi$, and $49\pi - 36\pi$. In total that is 28π inches squared. Since the board has an area of 64π, the other color has an area of $64\pi - 28\pi = 36\pi$. The probability that Merida's arrow will land in each section is $\frac{28\pi}{64\pi} = \frac{7}{16}$ and $\frac{36\pi}{64\pi} = \frac{9}{16}$. Since Merida can hit either color first, the overall probability is $\frac{7}{16} \times \frac{9}{16} \times 2 = \frac{63}{128}$. Therefore, $a + b = \boxed{191}$.

7. We convert 111000 to base 10 because we know that the binary number is of the form $111abc$. This is equal to $32 + 16 + 8 = 56$. We can add at most 7 to this number to keep the password 6-digits long (the largest 6-digit binary password possible is 63 in base 10). The largest prime number less than 63 is 61. In binary, $61_{10} = 111101_2$.Thus the last three digits are $\boxed{101}$.

8. The shortest possible value for XY is achieved when XY is parallel to AB, with a length of 5. The longest possible value for XY is achieved when $X = B$ and $Y = D$ or $X = C$ and $Y = A$, with a length of $\sqrt{5^2 + 9^2} = \sqrt{106}$. So, the longest integer length is 10. There are $10 - 5 + 1 = \boxed{6}$ integer values.

9. To play optimally, Team A wants to take out the players who have a higher chance of hitting their shots, so it is optimal for player 1 to shoot player 2. On Team B's turn, players 3 and 4 try to shoot at 1. The chance they both miss is $\frac{2}{3} \cdot \frac{3}{4} = \frac{1}{2}$. 1 then shoots 3, which is the optimal play, since 3 has a higher chance of hitting their target than 4. 4 then shoots at 1 and has a $\frac{3}{4}$ chance of missing. Then 1 can shoot 4, ending the game with team A winning. The chance this all occurs is
$$1 \cdot \frac{1}{2} \cdot 1 \cdot \frac{3}{4} \cdot 1 = \frac{3}{8},$$
so the answer is $3 + 8 = \boxed{11}$.

10. We will label the circle of radius 1 as O_4. Suppose the centers of O_1, O_2, O_3, and O_4 are at W, X, Y, and Z respectively, and the intersection points of O_3 with O_1, O_2, and O_4 are at L, M, and N respectively. Because O_1 and O_2 both have diameter $2r$, $\overline{LM}$ is the diameter of O_3. Because W is the center of O_1 and $\overline{YZ}$ is tangent to O_1, $m\angle WYZ = m\angle XYZ = 90°$. Hence, $\triangle WYZ$ is right. Applying the Pythagorean Theorem, we find that $r^2 + YZ^2 = (r+1)^2$. Moreover, YN and YL are radii of O_3, so $YN = YZ + 1 = 2r = YL$. Thus, we can substitute $YZ = 2r - 1$. All terms in the Pythagorean relationship are now expressed in r:

$$r^2 + (2r-1)^2 = (r+1)^2$$

$$r^2 + 4r^2 - 4r + 1 = r^2 + 2r + 1$$

$$4r^2 - 6r = 0$$

We select the positive value for r and find that $r = \frac{3}{2}$. The area of O_4 is thus $(\frac{3}{2})^2\pi = \frac{9}{4}\pi$, so $a + b = 9 + 4 = \boxed{13}$.

11. We can first prime factorize $8!$ as $2^7 \cdot 3^2 \cdot 5 \cdot 7$. We must choose exponents for a and b, for all four prime factors. First consider 2: The exponents of a and b, call them e and f, can be between 0 and 7, with $e \le f$. There are 8 ways to pick $e = f$ and $\binom{8}{2} = 28$ ways to pick $e < f$. In total, there are 8^2 ways to pick e and f, so the probability that $e \le f$ is $\frac{8+28}{64}$. Repeating this for primes 3, 5, and 7, we get probabilities of $\frac{6}{9}$, $\frac{3}{4}$, and $\frac{3}{4}$. In total, our probability is then $\frac{36}{64} \cdot \frac{6}{9} \cdot \frac{3}{4} \cdot \frac{3}{4} = \frac{27}{128}$. The final answer is then $\boxed{155}$.

12. If Nick swam for 10 minutes at 3 miles per hour, he swam $\frac{10}{60} \times 3 = \frac{1}{2}$ a mile. If Marj has swam for 25 minutes at 4 miles per hour, she swam $\frac{25}{60} \times 4 = \frac{5}{3}$ of a mile.
Call the distance Marj has left to swim after Marj and Nick meet m and the distance Nick has left to swim n. We know that $m + \frac{5}{3} - \frac{3}{4} = n + \frac{1}{2}$. Using power of a point, we also know that $\frac{5m}{3} = \frac{n}{2} \implies 10m = 3n \implies n = \frac{10m}{3}$.
Finally, solving the system of equations, we have that, $m + \frac{5}{3} - \frac{3}{4} = \frac{10m}{3} + \frac{1}{2} \implies \frac{7m}{3} = \frac{5}{12} \implies m = \frac{5}{28}$.
Therefore, Marj swims a total of $\frac{155}{84}$. This means $a + b = 155 + 84 = \boxed{239}$

13. The greatest integer less than 2000 with only even digits is 888. Similarly, the smallest integer greater than 2000 with only even digits is 2002. Therefore, the input 2000 could represent any integer in the interval $[\frac{2000+888}{2}, \frac{2000+2002}{2}]$, or $[1444, 2001]$.
Similarly, the greatest integer less than 600 with only even digits is 488, and the smallest integer greater than 600 with only even digits is 602. Therefore, the input 600 represents could be any integer in the interval $[\frac{600+488}{2}, \frac{600+602}{2}]$, or $[544, 601]$.
By the same reasoning, 40 could represent any integer in the interval $[\frac{40+28}{2}, \frac{40+42}{2}]$, or $[34, 41]$, and 8 could represent any integer in the interval $[7, 9]$.
Choosing 1444, 544, 34, and 7 as our intended inputs makes our displayed result as far from the intended result as possible. Thus, our maximum error is $(2000 - 1444) + (600 - 544) + (40 - 34) + (8 - 7) = 556 + 56 + 6 + 1 = \boxed{619}$.

14. Climbing n stairs can be divided into three cases depending on the number of stairs you climb on the last turn. Thus, if $S(n)$ is the number of ways to climb n stairs, then $S(n) = S(n-1) + S(n-2) + S(n-3)$. Note that $S(1) = 1, S(2) = 3, S(3) = 4$.

$$
\begin{aligned}
S(4) &= S(3) + S(2) + S(1) = 4 + 2 + 1 = 7 \\
S(5) &= S(4) + S(3) + S(2) = 7 + 4 + 2 = 13 \\
&\vdots \\
S(8) &= S(7) + S(6) + S(5) = 44 + 24 + 13 = 81 \\
S(9) &= S(8) + S(7) + S(6) = 81 + 44 + 24 = 149
\end{aligned}
$$

Thus, there are $\boxed{149}$ ways.

15.

$$S_{n+1} + S_n = a_{n+1}^2 \tag{1}$$
$$S_n + S_{n-1} = a_n^2 \tag{2}$$

Subtracting (2) from (1), we get

$$
\begin{aligned}
a_{n+1} + a_n &= a_{n+1}^2 - a_n^2 \\
a_{n+1} + a_n &= (a_{n+1} + a_n)(a_{n+1} - a_n)
\end{aligned}
$$

Since we know that $a_{n+1} + a_n \neq 0$, it must be that $a_{n+1} - a_n = 1$ for $n \geq 2$. From $S_2 + S_1 = a_2^2$, we get $a_1 + a_2 + a_1 = a_2^2 \implies a_s^2 - a_2 - 12 = 0 \implies (a_2 + 3)(a_2 - 4) = 0 \implies a_2 = 4$. Using $a_{n+1} - a_n = 1$ and $a_2 = 4$, we get $a_n = a_2 + (n - 2) = n + 2$ for $n \geq 2$ and $a_{20} = \boxed{22}$.

16. We can plug in $x = \lfloor x \rfloor + \{x\}$ where $\{x\}$ represents the fractional part of x. (In other words, $\{x\} = x - \lfloor x \rfloor$.) With this substitution, we get

$$
\begin{aligned}
f(x) &= \lfloor \lfloor x \rfloor^2 + 2\lfloor x \rfloor \{x\} + \{x\}^2 \rfloor + \lfloor x \rfloor^2 \\
&= 2\lfloor x \rfloor^2 + \lfloor 2\lfloor x \rfloor \{x\} + \{x\}^2 \rfloor.
\end{aligned}
$$

If $\lfloor x \rfloor = 0$, there is 1 possible value of k. If $\lfloor x \rfloor = 1$, there are 3 possible values. If $\lfloor x \rfloor = 2$, there are 5 possible values. In general, for $\lfloor x \rfloor = n$, there are $2n - 1$ values of k.

50 is one more than the sum of the first seven positive odd integers. We can either find this by adding the numbers up or by noticing that the sum of the first n odd numbers is n^2. The 50th smallest value of k is the smallest value of k when $\lfloor x \rfloor = 7$. From here, the desired value is easily obtained by plugging in $x = 7$ in $f(x)$, which gives $2 \cdot 7^2 = \boxed{98}$.