Joe Holbrook Memorial Math Competition

8th Grade Solutions

March 20, 2022

1. At each intersection of two streets, two equal obtuse angles and two equal acute angles are formed. Therefore, exactly half the corners are acute angles which is equivalent to $\boxed{50}$ percent.

2. There are 7 consonants and 3 vowels. Because the first and last letters are consonants, the 2nd and 4th letters must be vowels (as there must be at least one vowel between two consonants.) The middle letter can be either a vowel or a consonant for the word to still work, so there are $7 \cdot 7 \cdot 3 \cdot 3 \cdot 10 = \boxed{4410}$ options.

3. Denote Moana's speed as m and the wind's speed as w. Since distance is equal to speed times time. $21 = (m + w) \times 3$ and $21 = (m - w) \times 4$. If the system of equations is solved, it can be found that $w = \frac{7}{8}$. So, the answer is $7 + 8 = \boxed{15}$.

4. Firstly, we must find the areas of the two different colors. One color will be made of the portions of the rings with 1 ,3 ,5 , and 7 inch radii. The respective areas are π, $9\pi - 4\pi$, $25\pi - 16\pi$, and $49\pi - 36\pi$. In total that is 28π inches squared. Since the board has an area of 64π, the other color has an area of $64\pi - 28\pi = 36\pi$. The probability that Merida's arrow will land in each section is $\frac{28\pi}{64\pi} = \frac{7}{16}$ and $\frac{36\pi}{64\pi} = \frac{9}{16}$. Since Merida can hit either color first, the overall probability is $\frac{7}{16} \times \frac{9}{16} \times 2 = \frac{63}{128}$. Therefore, $a + b = \boxed{191}$.

5. Suppose that at a given time as the alarm is buzzing, m is the expected number of minutes until Aminah gets up (assuming she is still asleep). There is a probability of $\frac{1}{20}$ that she will get up at that moment, or in 0 minutes.

   Meanwhile, there is a probability of $\frac{19}{20}$ that she will hit the snooze button, in which case she will return to the original state where the alarm buzzes. Because the situation is identical, she will get up an expected m minutes after that. In this case, Aminah will get up in $m + 1$ minutes, where the 1 accounts for the minute she snoozed.

   Because these two cases account for the only paths Aminah can take, we conclude $m = \frac{1}{20} \cdot 0 + \frac{19}{20} \cdot (m + 1)$. Solving this equation, we find that $m = 19$. The case that we have worked with is equivalent to the initial situation where 0 minutes have passed, so $m = n = \boxed{19}$.

6. Intuitively, we know that the first time the hour hand and the minute hand will meet after midnight will be sometime after 1 am. Consider the angle the hour hand sweeps out in this interval. This angle is 2π less than the angle the minute hand sweeps out in the same interval, because the minute hand will have swept out the whole circle (i.e. swept out 2π radians) before catching up to the hour hand again.

   We know the angular velocity of the minute hand is $2\pi \frac{\text{radians}}{\text{hour}}$, and that of the hour hand is $\frac{2\pi}{12} \frac{\text{radians}}{\text{hour}} = \frac{\pi}{6} \frac{\text{radians}}{\text{hour}}$. Combining this with what was said in the previous paragraph, we have that

   $$\frac{\pi}{6} t = 2\pi t - 2\pi$$

   $$2\pi = \frac{11\pi}{6} t \to t = \frac{12}{11} \text{ hours}$$

   Notice that this value of t is not merely how long it is until the *next* time the hands meet. It is the length of time between *any* two instances where the hands meet! In other words, the second time the hands meet is $\frac{12}{11}$ hours after the first time, and the third time is $\frac{12}{11}$ hours after the second. Therefore, the third time the hands meet is $\frac{36}{11}$ hours, or about 3 hours and 16 minutes, after midnight.

   $h = 3, m = 16 \to 100h + m = \boxed{316}$

7. Reducing the given expression to a common denominator, we get

$$\left|\frac{r(r-s)+s(s-t)+t(t-r)}{rst} = \frac{r^2+s^2+t^2-(rs+st+rt)}{rst}\right|.$$

By Vieta's Formulas, $r+s+t=17$, $rs+st+rt=-105$, and $rst=-49$. We can obtain $r^2+s^2+t^2$ by taking $(r+s+t)^2-2(rs+rt+st)$, so the numerator is equivalent to $(r+s+t)^2-3(rs+rt+st)$ $=289+3(105)=604$. 604 is not divisible by 7 (the sole unique prime factor of 441), so the answer is $604+|-49|=604+49=\boxed{653}$.

8. If Nick swam for 10 minutes at 3 miles per hour, he swam $\frac{10}{60}\times 3=\frac{1}{2}$ a mile. If Marj has swam for 25 minutes at 4 miles per hour, she swam $\frac{25}{60}\times 4=\frac{5}{3}$ of a mile.

Call the distance Marj has left to swim after Marj and Nick meet m and the distance Nick has left to swim n. We know that $m+\frac{5}{3}-\frac{3}{4}=n+\frac{1}{2}$. Using power of a point, we also know that $\frac{5m}{3}=\frac{n}{2} \implies 10m=3n \implies n=\frac{10m}{3}$.

Finally, solving the system of equations, we have that, $m+\frac{5}{3}-\frac{3}{4}=\frac{10m}{3}+\frac{1}{2} \implies \frac{7m}{3}=\frac{5}{12} \implies m=\frac{5}{28}$. Therefore, Marj swims a total of $\frac{155}{84}$. This means $a+b=155+84=\boxed{239}$

9. If the exponents of the primes in the prime factorization of n are $e_1, e_2, \ldots, e_k$, then we want

$$\frac{3e_1+1}{e_1+1}\cdot\frac{3e_2+1}{e_2+1}\cdots\frac{3e_k+1}{e_k+1}=7.$$

We can see that that $\frac{3e_i+1}{e_i+1}<3$. Further, when $e_i=1$, $\frac{3e_i+1}{e_i+1}=2$, so $2\le\frac{3e_i+1}{e_i+1}<3$. So, we must have $k=1$ or 2 (if we have at least 3, then our expression is at least $2^3=8$). If $k=1$, then $3e_1+1=7(e_1+1)$ yields no solutions. If $k=2$, then $(3e_1+1)(3e_2+1)=7(e_1+1)(e_2+1)$. Expanding both sides and rearranging, we get $2e_1e_2-4e_1-4e_2=6$. If we divide both sides by 2 and then add 4 to both sides, this allows us to factor (Simon's Favorite Factoring Trick) as $(e_1-2)(e_2-2)=7$. So, the only solutions are $e_1-2, e_2-2=1,7$. This gives $e_1=3, e_2=9$ (or vice-versa), and so the smallest possible value of n is $2^9\cdot 3^3=\boxed{13824}$.

10. Label the midpoint of BC as M. Also, draw the altitude from point A to BC and mark the intersection as H. From the Pythagorean theorem, we know that $11^2-CH^2=AH^2$ and that $9^2-(10-CH)^2=AH^2$. Therefore, $11^2-CH^2=9^2-(10-CH^2)$. Now we can solve for CH: $121-CH^2=-19+20CH-CH^2$, $CH=7$. Since $MC=10/2=5$, that means $HM=2$. We also know that $AH=6\sqrt{2}$. Therefore, we can solve for BC^2: $BC^2=(2^2+(6\sqrt{2})^2)^2=\boxed{76}$

11. Each set of four colors can correspond to exactly two different paintings. To see this, let's label the colors 1 to 5; consider a tetraherdon painted with colors 1, 2, 3, 4. Imagine placing the tetrahedron on the table so that corner 4 is pointed up. Corners 1, 2, 3 can either be clockwise or counterclockwise, and no rotation can change between a clockwise or counterclockwise tetrahedron; this is called chirality. There are $\binom{5}{4}=5$ different possible sets of four paints to use, and each set of four paints leads to two paintings, so the answer is $2\cdot 5=10$.

12. We would like to find each integer n between 1 and 2022 such that $225^n+227^n\equiv 0 \pmod{791}$. We note that $226=2\cdot 113$, $791=7\cdot 113$, and 113 is prime. By the Chinese Remainder Theorem, we can find the remainder of this expression $\pmod{791}$ by considering its residue in mods 7 and 113. For n to be divisible by both 7 and 113, each residue must be 0.

We first consider $\pmod{113}$. We can express each n in the form $(2\cdot 113+1)^n+(2\cdot 113-1)^n\equiv 1^n+(-1)^n$, which is $0 \pmod{113}$ for odd n and $2 \pmod{113}$ for even n. Therefore, each n is odd.

We now consider $\pmod{7}$. We have $(32\cdot 7+1)^n+(32\cdot 7+3)^n\equiv 3^n+1^n\equiv 0 \pmod{7}$. By inspection, $n\equiv 3 \pmod{6}$.

We have found two conditions for n: it must be odd and congruent to $3 \pmod{6}$. Because every number congruent to $3 \pmod{6}$ is odd, we only need to consider the second condition. As $2022=6\cdot 337$, there are $\boxed{337}$ such numbers between 1 and 2022.

13. First, since AB is a diameter, $\angle ACB$ is a right angle. Thus, by the Pythagorean Theorem, $CB = \sqrt{10^2 - (6+2)^2} = 6$. Then, since $\angle ECB = \angle EDA = 90^\circ$ and $\angle CEB = \angle DEA$ as they are vertical angles, $\triangle CEB \sim \triangle DEA$ by AA. So, $\frac{DA}{DE} = \frac{6}{2} = 3$, and we can let $DE = k$, $DA = 3k$. Further, by Power of a Point (or just using the similar triangles again), $BE = \frac{2 \cdot 6}{k} = \frac{12}{k}$. Now, by the Pythagorean Theorem,
$$DA^2 + DB^2 = (3k)^2 + \left(k + \frac{12}{k}\right)^2 = AB^2 = 100.$$
Expanding and rearranging, this is a quadratic equation in k^2, which we can solve to find $k^2 = \frac{18}{5}$. Finally, $AD^2 = 9k^2 = \frac{162}{5}$, and the answer is $\boxed{167}$.

14. Suppose Abuna, Balthazzar, Camillus, Dakari, Eido, and Fen get a, b, c, d, e, f apples respectively. Since 16 is 1 mod 3, one of d, e, f has to be 2 mod 3, while the other two are 1 mod 3. There are three ways to pick which of the three is 2 mod 3. Assuming for now that it is d, note that we can write $a = 3a'$, $b = 3b'$, $c = 3c'$, $d = 3d' + 2$, $e = 3e' + 1$, and $f = 3f' + 1$, for nonnegative integers a' through f'. Since $a + b + c + d + e + f = 16$, we can substitute this in, rearrange, and divide by through by 3 to obtain $a' + b' + c' + d' + e' + f' = 4$. By Stars and Bars, there are $\binom{9}{4} = 126$ solutions to this equation. Finally, remembering that we have to multiply by 3, to pick which of d, e, f is 2 mod 3, our final answer is $\boxed{378}$.

15. Let $\gcd(a, b) = g$. Then, we can express a and b as $a = gx$ and $b = gy$, where x and y must be relatively prime - otherwise, g would be larger. Also note that $\operatorname{lcm}(a, b)$ is then gxy. So, we must have
$$gxy = g^g \implies xy = g^{g-1}.$$
Now, if g has only prime in its prime factorization, then one of x, y would have to be 1, in order to keep them relatively prime. However, this would mean that one of a, b is a multiple of the other. So, we must consider a g with at least 2 prime divisors. The smallest possibility is $g = 6$. Then, we would have to split 6^5 as $x = 2^5$ and $y = 3^5$ (or vice versa). Our answer is then $(2^5 + 3^5)g = 275 \cdot 6 = \boxed{1650}$.

16. Suppose the two numbers are x and y at some point. If $x = y$, then the maximum will necessarily increase the next minute, while the minimum will not. If $x \neq y$, then the maximum and minimum have equal chance of changing (but they cannot both change). Note that the latter case will vanish in $B - A$, as it will have the same contribution in both A and B.

So, we can see that $B - A$ is simply the expected number of times $x = y$ in the first 9 minutes (it doesn't matter if they are equal in the 10th minute, since there is no 11th minute for the maximum to increase in). The probability that $x = y$ after $2k$ minutes (obviously, we cannot have $x = y$ when an odd number of minutes have passed) is
$$\frac{\binom{2k}{k}}{2^{2k}}.$$
We must choose k of the $2k$ minutes to be increases for x, while the other k are increases for y. The 2^{2k} in the denominator comes from the fact that there are 2^{2k} different possibilities for $2k$ minutes (there are two possibilities every minute). Thus,
$$B - A = 1 + \frac{\binom{2}{1}}{2^2} + \frac{\binom{4}{2}}{2^4} + \frac{\binom{6}{3}}{2^6} + \frac{\binom{8}{4}}{2^8} = \frac{315}{128},$$
where the five terms are the probability that $x = y$ after 0, 2, 4, 6, and 8 minutes respectively. The answer is $\boxed{443}$.