Joe Holbrook Memorial Math Competition

5th Grade Solutions

March 20, 2022

1. The problem allows us to set up the equation $0.24x = 0.30y$. We are asked to find what percent of x is y, so we need to isolate y. Dividing by 0.30 on both sides yields $y = \frac{0.24}{0.30}x = 0.80x$, meaning our answer is $\boxed{80}$.

2. Tasha has 7 choices for which place to go to first. From there, she has 6 choices, from there she has 5 choices, etc. In other words, the number of choices for where Tasha can go next keeps decreasing by 1. Therefore, the total paths she can take is $7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = \boxed{5040}$.

3. For them to both make 1200 cutouts, Timmy takes $1200/10 = 120$ hours to finish, and Shawn takes $1200/15 = 80$. The answer is $\boxed{40}$.

4. This problem is asking for an arithmetic series. The sequence is

$$5, (5 + 4 \times 1), (5 + 4 \times 2), (5 + 4 \times 3), \ldots (5 + 4 \times 10).$$

Notice that $5 + (5 + 4 \times 1) + (5 + 4 \times 2) + (5 + 4 \times 3) \ldots (5 + 4 \times 10) = (5 \times 11) + (4 \times (1 + 2 + 3 + \ldots 10)) = 55 + 4 \times 55 = \boxed{275}$ seashells.

5. $293 \div 14 = 20$ R 13, so Eshaan needs $20 + 1 = \boxed{21}$ days to listen to at least 293 songs.

6. Call the length of the picture frame l and the width of the picture frame w. We are given that $2l + 2w = 66$ and that $l = 2w$. Combining these two equations, we can find that $w = 11$ and $l = 22$. The length of the picture itself is $l - 4 = \boxed{18}$ inches.

7. Because a problem's difficulty is equal to the number of minutes it takes to write a solution for it, we can deduce that Lance's problems' average time to write solutions for is 6 minutes. So it'll take $12 \times 6 = \boxed{72}$ minutes for him to write all of his solutions.

8. To find the total number of syllables, multiply the number of options for the consonant by the number of options for the vowel. Since $14 * 18 = 252$ and there are four more consonants than vowels, we know the number of vowels in Jacob's alphabet must be $\boxed{14}$.

9. Rose should apply the 20% off coupon first because the base value it is being applied to is greater, meaning a greater value is being deducted. This coupon reduces the price to $100 \times 0.8 = 80$ dollars. After the second coupon, Rose only has to pay $80 - 10 = \boxed{70}$ dollars.

10. We can simply try each prime - we find that $3^2 + 4$, $5^2 + 4$, and $7^2 + 4$ are prime, but $\boxed{11}^2 + 4 = 125$ is not.

11. If it rains today, then there is a $\frac{2}{5} \cdot \frac{1}{4} = \frac{2}{20}$ chance that it will rain tomorrow. If it does not rain today, then there is a $\frac{3}{5} \cdot \frac{3}{4} = \frac{9}{20}$ chance it will rain tomorrow. Summing the probabilities yields a total probability of $\frac{11}{20}$. Therefore, the answer is $11 + 20 = \boxed{31}$.

12. If a has 9 positive multiples less than 100, then $\frac{100}{a}$ must be equal to 9 with some remainder or 10 with no remainder (since we are only counting multiples LESS than 100, not equal to 100). After testing some numbers, we see that $100/10 = 10$ w/no remainder and $100/11 = 9$ w/remainder. So, a can only be 10 or 11. If b has 19 positive multiples less than 100, then $\frac{100}{a}$ must be equal to 19 with some remainder or 20 with no remainder. After testing numbers, we see that $100/5 = 20$ w/no remainder, so b can only be 5. Thus, ab is either 50 or 55. However, in either case, there is only 1 multiple of ab strictly less than 100 and that is $ab \cdot 1$. Therefore, $\boxed{1}$ is our answer.

13. This situation can be represented with the recurrence relation $a_n = 2 * a_{n-1} - 3$, where a_n represents the amount of cupcakes I have on day n and $a_1 = 4$. Notice that the amount of cupcakes that I have follows the pattern that on the nth day, I will have $2^{n-1} + 3$ cupcakes. This follows from even the first day and can be more easily observed after listing terms. Thus, our answer is $2^{10-1} + 3 = \boxed{515}$.

14. It takes $\frac{450}{60} \cdot 60 = 450$ minutes to drive (time is equal to distance divided by speed and there are 60 minutes in an hour). Similarly, it takes $\frac{400}{240} \cdot 60 + 120 = 220$ minutes to fly (the extra 120 is from the two hours of arriving/boarding). The answer is therefore $450 - 220 = \boxed{230}$ minutes.

15. The unit digit of powers of 2 repeat in the order $2, 4, 8, 6$, while powers of 5 always have a unit digit of 5. Because 2022 leaves a remainder of 2 when divided by 4, the units digit of 2^{2022} is 4, and the units digit of 5^{2022} is 5. Thus, the unit digit of the sum will be $4 + 5 = \boxed{9}$.

16. Let S be the number of snakes in the zoo, C be the number of chickens in the zoo, and P be the number of parrots in the zoo. Since there are 24 total animals, it is known that $S + C + P = 24$. However, because only chickens and parrots have legs, $38 = 2C + 2P$, which means that there are 19 chickens and parrots combined. Therefore, there must also be $\boxed{5}$ snakes.

17. We know that if a convex polygon has n sides, then its interior angle sum is given by $(n - 2) * 180^\circ$. We also know that because it is convex, each angle is less than 180 degrees. That means we only need to find the closest multiple of 180 greater than 2022. $180 \times 12 = 2160$, so our answer is $180 \times 12 - 2022 = \boxed{138}$ degrees.

18. We note that for any fishy number n, n cannot be a one digit number: if so, then because its only digit was prime, n would itself be prime, contradicting the problem. Therefore, $n \geq 10$. Both of n's digits must be elements of $\{0, 4, 6, 8, 9\}$. Since $n > 2$, n cannot be even as this would make n composite, so the units digit of n must be 9, making the only possible fishy numbers $49, 69, 89$, and 99. Because $49 = 7^2, 69 = 3 * 23$, and $99 = 3^2 * 11$, the prime number 89 is the only fishy number, so the sum is $\boxed{89}$.

19. Since the line connecting Austin and Pablo is parallel to the line connecting Tyrone and Tasha, the line connecting Austin and Tasha is a transversal. Since the angle from Tyrone, to Pablo, to Austin is equal to the angle from Austin, to Tasha, to Tyrone, the line between Austin and Tasha intersects the line between Pablo and Tyrone at the center of the crater. This means that the angle from Pablo, to the center of the crater, to Tasha is 80 degrees. From there, since the the line between Austin and Tasha intersects the center of the crater, the angle from Austin, to the center of the crater, to Pablo is $180 - 80 = 100$. This means that the arc measure between Austin and Pablo is $\boxed{100}$ degrees.

20. We know that Cathy spent a total of $5 \times 50 = 250$ cents. Let x being the number of dimes, and y being the number of quarters spent. With the given information, we know that the following equations are true:

$$250 = 10x + 25y$$

$$x + y = 16$$

Solving this system of equation, we get that $x = 10$ and $y = 6$. So, Cathy spent $\boxed{6}$ quarters.

21. Merlin can either pull out two cards of the same suit or three cards of the same suit. Let's first consider the case in which Merlin pulls out two cards of the same suit. There are three ways Merlin can do this (just choose one of the three cards to be different). In any of these cases, once Merlin draws the different card, he has a $\frac{12}{51}$ chance of getting a card of the same suit and a $\frac{31}{50}$ chance of getting a card of a different suit. This is a $1 \cdot \frac{12}{51} \cdot \frac{39}{50} = \frac{468}{2550}$ probability. In total, then, there is a $\frac{468}{2550} \times 3 = \frac{1404}{2550}$ probability of Merlin getting 2 cards of the same suit.

Now, for Merlin to get 3 cards of the same suit, he can first pull out any of the 52 cards. Once he has this card, he must pull out one of the 12 remaining cards of the same suit out of the deck of 51 remaining cards and then pull out one of the 11 remaining cards of the same suit out of the deck of 50 remaining cards. This is a $1 \cdot \frac{12}{51} \cdot \frac{11}{50} = \frac{132}{2550}$ probability.

$\frac{1404}{2550} + \frac{132}{2550} = \frac{1536}{2550}$ probability of Merlin pulling out at least to cards of the same suit. This simplifies to $\frac{256}{425}$ when the numerator and denominator are divided by 6, so $m = 256$, $n = 425$, and $m + n = \boxed{681}$.

22. This year is not a leap year, so seven months have 31 days (January, March, May, July, August, October, and December), four months have 30 days (April, June, September, November), and February has 28 days. There are 9 primes less than or equal to 28 (2, 3, 5, 7, 11, 13, 17, 19, and 23), 10 primes less than or equal to 30 (the same nine as before as well as 29), and 11 primes less than or equal to 31 (those ten primes and 31 itself). So, the answer is $7 \cdot 11 + 4 \cdot 10 + 9 = 77 + 40 + 9 = \boxed{126}$.

23. As the woman walks in a circle, the point on her hat closest to the center is $6 - 4 = 2$ feet away from the center. That means there will be a circle with radius 2 feet that is not covered by the woman's hat. The area of that circle is $2^2 \times \pi = 4\pi$ feet squared. The point on her hat farthest from the center is $6 + 4 = 10$ feet away. This is yet another circle, this time of radius 10 feet with area $10^2 \times \pi = 100\pi$. The area of dead corn is $100\pi - 4\pi = 96\pi$, so our answer is $\boxed{96}$.

24. Substituting $y = 4^x$ gives us the equation $16^x - 4^{x+\frac{3}{2}} = y^2 - 8y - 128 = (y - 16)(y + 8) = 0$. Solving for $y = 16 = 4^x$, we see that an integer solution is $x = \boxed{2}$.

25. The president and vice president can be chosen 100(99) ways. From the remaining 98 people, they can be chosen 2^{98} ways. Thus the total number of possibilities is $(100)(99)(2^{98})$. The number of *ideal* possibilities are $2 \cdot (2^{98})$ because there are 2 ways to choose from Andrea and Anthony, and 2^98 ways to choose the rest of the people. Thus, the probability is $\frac{2 \cdot (2^{98})}{(100)(99)(2^{98})} = \frac{1}{50 \cdot 99} = \frac{1}{4950}$. Thus $m + n = 1 + 4950 = \boxed{4951}$.

26. If $\overline{DE}$ is parallel to $\overline{BC}$, then $\angle ABC \cong \angle ADE$ and $\angle ACB \cong \angle AED$ by the properties of the transversals $\overline{AB}$ and $\overline{AC}$ through the parallel lines. Since $\angle BAC \cong \angle DAE$ by reflexivity, all three angles of triangle ABC are congruent to their corresponding angles on triangle ADE which means that $\Delta ABC \sim \Delta ADE$. Now, if $\frac{AB}{DB} = 5$, then $DB = \frac{1}{5}(AB)$. And, since $AD + DB = AB$, this means that $AD + \frac{1}{5}(AB) = AB$, so $AD = \frac{4}{5}(AB)$. Since $\Delta ABC \sim \Delta ADE$, this ratio between sides must be the same for all pairs of corresponding sides, so each side of ΔADE is $\frac{4}{5}$ of its corresponding side on ΔABC. This means that the area of ΔADE is $(\frac{4}{5})^2$ of the area of ΔABC. Thus, the area of $\Delta ADE = \frac{16}{25} \cdot 50 = \boxed{32}$ units^2.

27. Let O be the center of the semicircle. By symmetry, $OC = OD = \frac{6}{2} = 3$, and by the Pythagorean Theorem, the radius OB equals $\sqrt{7^2 + 3^2} = \sqrt{58}$. The area of the semicircle is therefore $\frac{1}{2}\left(\sqrt{58}\right)^2 \pi = 29\pi$, so $a = \boxed{29}$.

28. Depending on how x compares to 5 and 8, the median must be 5, 8, or x. This means the mean, equal to $\frac{57 + x}{9}$, must be 6, 9, or $x + 1$. Solving for x, we get possible solutions of -3, 24, and 6, respectively; all of these end up working so the answer is $-3 + 24 + 6 = \boxed{27}$.

29. Let R represent a red button, O represent an orange button, and Y represent a yellow button. The total number of ways to arrange $RROOYY$ is $\frac{6!}{2!2!2!} = 90$, because there are 2 buttons of each color. Then, we can use PIE to find the total number of ways there will be at least one ROY. Using ROY as a single unit, there are 4! ways to arrange the $[ROY]$, R, O, Y. However, we have counted $ROYROY$ twice, so we subtract one of them to get rid of the over counting. Thus, the fraction is $\frac{4! - 1}{90} = \frac{23}{90}$, so the answer is $90 - 23 = \boxed{67}$.

30. The probability that k heads are flipped is probabilities as $\frac{3}{4}^k \frac{1}{4}^{5-k} \binom{5}{k}$ for k from 0 through 5. The fractions represents the likelihood of getting k heads and $5 - k$ tails and the binomial coefficient represents the amount of ways to arrange them. We want to find the sum of this expression when $k = 3$ and $k = 5$, which is $\frac{3}{4}^3 \frac{1}{4}^2 \binom{5}{3} + \frac{3}{4}^5 \frac{1}{4}^0 \binom{5}{5} = \frac{513}{1024}$, meaning the probability that Even-Steven wins is $1 - \frac{513}{1024} = \frac{511}{1024}$, meaning the positive difference is $\frac{2}{1024} = \frac{1}{512}$. Therefore, our answer is $\boxed{513}$.

31. The prime numbers from 1 to 6 are $2, 3$, and 5. The only ways to add 3 of them to get a prime number are $2 + 2 + 3 = 7$, $3 + 3 + 5 = 11$, $3 + 5 + 5 = 13$. However, since order matters here, all options have 3 permutations, making for a total of 9 possible ways for the sum to be prime. So, the number of ways to have all 3 numbers be prime but not their sum is $3^3 - 9 = 18$. Our answer is then $\frac{18}{6^3} = \frac{1}{12} \implies \boxed{13}$.

32. This problem is based on Bezout's Lemma, which states that there exists a linear combination of 4 and 7 such that they combine to 1. In essence, the healer needs to give the Hydra enough heads such that it has exactly 1 more than a multiple of 7. We can see that if the healer gives the Hydra 16 heads, the Hydraw will have $36 = 7 * 5 + 1$. This will take 4 actions. Finally, the knight will take off 35 of its heads with 5 actions. This is the least amount of actions, as the inverse of 4 modulo 7 is 2, and since the Hydra already has 20 heads, we must get to 28 heads, then 36 heads, and then subtract down. This means our amount of actions must be $4 + 5 = \boxed{9}$.

33. Since $32 = 2^5$ and $9 = 3^2$, $2^{5x} = 3^{2y}$. Since $243 = 3^5$, $z^x = 3^{5y}$. So, $z^x = 2^{25x/2}$, with $z = 2^{25/2}$. So, $\sqrt[5]{z^2} = \boxed{32}$.

34. Let's first focus on the probability of Terri winning exactly two matches in her division. We can use Binomial Probability to compute this. There are $\binom{4}{2}$ ways to choose the 2 games that she will win, and she needs to win exactly 2 and lose exactly 2, so she has a probability of $(0.80)^2 * (0.20)^2$. Hence, Terri's probability of winning exactly two matches is $\binom{4}{2}*(0.80)^2*(0.20)^2 = 6*(\frac{8}{10})^2*(\frac{2}{10})^2 = 6*\frac{64}{100}*\frac{4}{100} = \frac{96}{625}$. Now, we can calculate Tate's probability of winning two matches in his division. There are $\binom{3}{2}$ ways to choose the 2 games that he will win, and he has a probability of $(0.50)^2 * (0.50)$ of winning exactly 2 and losing exactly 1. So, Tate's probability of winning exactly two matches is $\binom{3}{2} * (0.50)^3 = 3 * (\frac{1}{2})^3 = 3 * \frac{1}{8} = \frac{3}{8}$. Finally, because both events are independent of each other and because they both have to occur, we multiply Terri and Tate's probabilities: $\frac{96}{625} * \frac{3}{8} = \frac{36}{625}$. Since 36 and 625 share no common factors greater than 1, $\frac{m}{n} = \frac{36}{625}$, and therefore, $m + n = 36 + 625 = \boxed{661}$.

35. For this problem, every piece of information must be use to narrow down the scope of possibilities to a single number. First, note the fact that this is a birth-date so the number is at most 31. The list of all of the primes that are at most 31 include:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31$$

Then, we can use the fact that this number can be represented as the sum of two non-negative squares. Note that all primes in the form $4k + 3$ for $k \in \{0, 1, 2, ...\}$ cannot be written as a sum of two squares, as squaring all of the remainders mod 4 will only yield remainders of 0 and 1, meaning no sum of those two numbers can create a remainder of 3. This allows us to remove $3, 7, 11, 19, 23$ and 31, yielding the new list

$$2, 5, 13, 17, 29$$

It is easy to check each number to see that they satisfy this property $(1^2+1^2, 1^2+2^2, 2^2+3^2, 1^2+4^2, 2^2+5^2$ respectively). Then to use the last property given, we can list out the sum of the digits, being $2, 5, 4, 8,$ and 11 respectively. A number does not have an even number of divisors when it is a perfect square, so only $\boxed{13}$ satisfies all the properties.

36. We observe that n is a threehugger if and only if n is the square of a prime. (If it is of this form, then n will be divisible by 1, $\sqrt{n}$, n, and no other integers. Meanwhile, if n is not a square it has an even number of divisors, and if it is the square of a composite number it has more than 3 divisors.) Hence, we must count the number of primes p such that $333 \leq p^2 \leq 2023$. Because $324 = 18^2$ and $2025 = 45^2$, this will be the number of primes between 18 and 45 exclusive, which is $\boxed{7}$ $(19, 23, 29, 21, 37, 41, 43)$.

37. Let the cube have side length c and the tetrahedron have side length t. Then, $c^3 = \frac{t^3}{6\sqrt{2}}$ or

$$\frac{t^3}{c^3} = 6\sqrt{2}$$

$$\frac{t^6}{c^6} = 72.$$

The surface area of the cube is $6c^2$ and the surface area of the tetrahedron is $4 \cdot \frac{\sqrt{3}}{4}t^2 = \sqrt{3}t^2$. Therefore, the ratio is

$$(\frac{\sqrt{3}t^2}{6c^2})^3 = \frac{3\sqrt{3}t^6}{216c^6} = \frac{72 \times 3\sqrt{3}}{216} = \sqrt{3}.$$

Hence, the answer is $\boxed{3}$.

38. In base 6, these integers range from 1 to 5555. If we add leading zeros such that they each have four digits, we can basically consider these numbers as strings of four digits, each ranging from 0 to 5, but 0000 doesn't count. Also, adding 0s naturally does not change the number of non-zero digits. If there are two non-zero digits, we have $\binom{4}{2} \cdot 5^2 = 150$ choices ($\binom{4}{2}$ is choosing the positions for the non-zero digits, and 5^2 is choosing what they are). Similarly, there are $\binom{4}{4} \cdot 5^4 = 625$ integers with four non-zero digits, so the total is $150 + 625 = \boxed{775}$.

39. If we consider the integers from 0 to 31, the largest binary representation would be 1111. Each digit here is $\frac{1}{2}$ of the time equal to 1, and the other $\frac{1}{2}$ equal to 0. Thus, if we sum the digits individually, we have that for 16 of the numbers, the thousand digit is 1, and the other 16, it will be 0. Thus, the sum for this digit will be $16(1000) + 16(0) = 16000$. If we continue this reasoning for the remaining numbers, we get $16000 + 1600 + 160 + 16 = 17776$. However, since we only want to sum from 1 to 29 in base 2, we must exclude the binary representations of 30 and 31 from this sum. These are 1110 and 1111 respectively, so subtracting them out our final answer is $\boxed{15555}$.

40. Pollo and Chicken each can roam in a quarter circle of radius 4. We can draw this, and label Pollo's vertex of the pen as A and Chicken's vertex of the pen as B. Call the two intersection points of these quarter circles as P and Q. We know that the hypotenuse of the triangle of sides 4 and $4\sqrt{2}$ is $4\sqrt{3}$. Half of this will give the height of each of the two triangles constructed when we connect AQP and BQP. These will have height $2\sqrt{3}$, and since the radius is 4, the base will be 2. These ratios tell us that this is a $30-60-90$ triangle.

    Thus, the sector of 60° of a circle of radius 4 is $\frac{4^2\pi}{6} = \frac{8\pi}{3}$. Subtracting out the equilateral triangle, we get half the region in which their roaming areas overlap. This will be $\frac{8\pi}{3} - 4\sqrt{3}$, and multiplying by 2, we get $\frac{16\pi}{3} - 8\sqrt{3}$.

    Since the area in which they can each individually roam is $\frac{4^2\pi}{4} = 4\pi$, we can divide this and get $\frac{4}{3} - \frac{2\sqrt{3}}{\pi}$. Thus our answer is $12 + 4 + 3 = \boxed{19}$.