Answer Sheet

Name: ______________________ **ID**: ____________

1.	11.	21.	31.
2.	12.	22.	32.
3.	13.	23.	33.
4.	14.	24.	34.
5.	15.	25.	35.
6.	16.	26.	36.
7.	17.	27.	37.
8.	18.	28.	38.
9.	19.	29.	39.
10.	20.	30.	40.

FOR GRADER USE ONLY:

Score 1	Score 2	Score 3	Score 4

Total Score:

Joe Holbrook Memorial Math Competition

7th Grade

October 16, 2022

General Rules

- You will have **75 minutes** to solve **40 questions**. Your score is the number of correct answers.
- Only answers recorded on the answer sheet will be graded.
- This is an individual test. Anyone caught communicating with another student will be removed from the exam and their score will be disqualified.
- Scores will be posted on the website. Please do not forget your ID number, as that will be the sole means of identification for the scores.
- You may use the following aids:
  - Pencil or other writing utensil
  - Eraser
  - Blank scrap paper
- You may not use the following aids:
  - The Internet
  - Books or other written sources
  - Other people
  - Calculator or other computing device
  - Compass
  - Protractor
  - Ruler or straightedge

Other Notes

- All answers are integers. Make sure you do not make any mistakes when writing your answers, as you will not be given credit if you do so.
- You do not need to write units in your answers.
- Ties will be broken by the number of correct responses to questions 31 through 40. Further ties will be broken by the number of correct responses in the last five questions.

1. Albert goes to a store and uses 20 percent of his money in the wallet. He then goes to another store to spend 50% of the remaining money in his wallet. Finally, he gives 10 dollars to his brother and realizes that he has no money left in his wallet. How many dollars did Albert start with?

2. King Henry VIII challenges his court to guess his favorite natural number. A wise nobleman is able to correctly determine his favorite natural number after asking three questions, all of which the king answers correctly: is it 1-digit, is it prime, and is it odd? What is the king's favorite natural number?

3. Oh no! Lance forgot about Eshaan's birthday! As an apology, Lance buys Eshaan a concert ticket; the ticket costs \$115. If Lance has \$240 in his bank account, how much money (in dollars) will he have after he buys the ticket?

4. Phoenix has a 13" $\times$ 4" sheet of chocolate cake which she wants to cut into 2" $\times$ 2" pieces. At most how many pieces can Phoenix get?

5. Riley is making gift bags with pencils and candy for her birthday party. If she wants each bag to have the same contents and she has 399 pencils and 168 candies, what is the greatest number of gift bags that Riley could make with no items leftover?

6. The classroom bookshelf has two history books, three English books, four math books, and five science books. If I take books off the shelf at random, how many do I need to take to guarantee that I have two books of the same subject?

7. Let a be a positive integer such that there are 9 positive multiples of a less than 100. Let b be a positive integer such that there are 19 positive multiples of b less than 100. How many positive multiples of ab are less than 100?

8. Justin wants to test if his new computer is working. He selects an arbitrary real number k from 0 to 10 inclusive, and asks his computer to find the solution to the equation $3x + k = 0$. The probability that the output of the computer is less than $-\frac{2}{5}$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

9. Suppose there exists a sequence that starts with the term 4210, and where each successive term is computed by summing the cubes of the digits of the previous term. What is the 2022nd term of this sequence?

10. Tyron has an apple tree with 40 apples. The Apple Man comes along and tells Tyron three true statements:

    - The number of rotten apples in the tree is a perfect square.
    - The number of non-rotten apples shares a prime factor with 10.
    - There are an odd number of non-rotten apples.

    How many apples in the apple tree are not rotten?

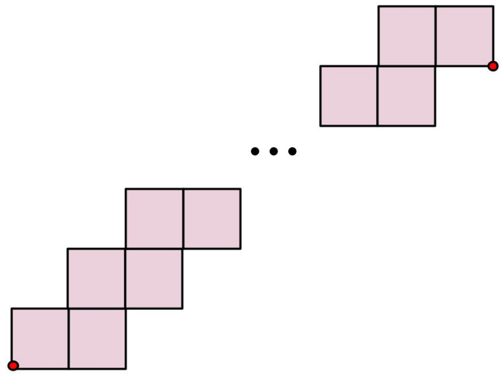
11. Austin has a 15-foot-long log that he wants to use to make the base of a wooden hut. He wants the hut's base to be a scalene triangle, and he does not have to use the whole log. How many non-congruent triangles can Austin create if all of their side lengths are integers?

12. In the 2D plane, the surface bounded by the lines $x = 0, x = 4, y = 2, y = -2$ is rotated a full revolution about the x-axis. Find the closest integer to the surface area of the 3D figure created.

13. One day, the Soda Shed sold 253 cans of juice to 100 consumers, and every consumer bought at least one can of juice. What is the maximum possible median number of cans of soda bought per consumer on that day?

14. Suppose A and B are digits with $1 \leq A, B \leq 9$. How many ordered pairs (A, B) are there so that AB is divisible by BA? (AB is the two-digit number with tens digit A and units digit B.)

15. The staircase below consists of eight layers of two squares, each layer shifted one unit right of the one below it. A line is drawn to connect the rightmost bottom point and leftmost point bottom (red). How many squares of the staircase does this line touch?



16. Pablo and Uniqua are each holding the end of a jump rope. They swing the rope so that it is in the shape of a parabola. When the jump rope reaches its maximum height, it follows the graph of $y = -\frac{1}{20}x^2 + x + 5$, and when it reaches its minimum height, it follows the graph of $y = \frac{1}{20}x^2 - x + 5$. Both graphs are created in relation to the ground being the x-axis. What is the difference between the maximum and minimum height of the jump rope?

17. Mr. Nelson writes down two positive integers, A and B, on a whiteboard and asks Matt what the sum of the two numbers are. Matt says that the sum is $94,395$, but it turns out that Matt misread every 1 as a 7 and every 7 as a 1 (his arithmetic is otherwise perfect). What is the greatest possible correct sum of A and B?

18. Alice and Bob take 3 hours to mow the lawn together, but Alice can do it alone in 5 hours. Assuming that each person always works at a constant speed, how many minutes does it take Bob to mow the lawn?

19. Let $x_1, x_2, \cdots, x_n$ denote some permutation of the numbers $1, 2, \cdots, n$. For how many odd integers $n \leq 2022$ does there exist such a permutation $x_1, x_2, \cdots, x_n$ such that the product $(x_1 - 1)(x_2 - 2) \cdots (x_n - n)$ is odd?

20. A function f has the property that $f(ab) = f(a) + f(b)$ and that $f(10) = 1$. If x such that $f(x) = 5/6$ can be written as $a^{\frac{b}{c}}$ where b and c are in lowest terms and $a > 1$, find abc.

21. Find the sum of all positive integers N such that the second largest factor of N is 15 times its second smallest factor.

22. Let $s(n)$ be the sum of digits of n. What is the smallest positive integer n such that $s(s(n)) = 11$?

23. Suppose that x is a 3 digit number (with the first digit non zero). We say x has a "nice average" if it is the average of the six numbers formed by the permutations of its digits. An example of a number which has a nice average is 629, because 629 is the average of the numbers $629, 692, 269, 296, 926$, and 962. The sum of all 3 digit numbers which have a "nice average" is S. Compute the remainder when S is divided by 37.

24. Kathy is randomly sewing 6 buttons onto her shirt in a vertical line. She has three colors of buttons, red, orange, and yellow, and two of each. The probability that there will be at least one consecutive ordering of red, orange, yellow, on her shirt from top to bottom, can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m - n$.

25. A healer and knight are battling a hydra with with 20 heads. The healer can give the hydra exactly 4 heads each action. The knight can take exactly 7 heads off each action. The hydra will only be defeated if it has exactly 1 head left. If it has any less, it will immediately grow back to its original 20 heads state. What is the least amount of actions taken by the two to defeat the hydra?

26. Two circles of radius 6 have centers O_1 and O_2 such that $O_1O_2 = 6\sqrt{3}$. If the area of the region contained in both circles can be expressed as $a\pi + b\sqrt{c}$ for integers a, b, and c, with c nonnegative and squarefree, find $a + b + c$.

27. 6 doughnuts—2 jelly-filled, 2 chocolate, and 2 glazed—are to be divided up among Adisa, Batu, and Catalina so that everyone receives exactly 2 doughnuts. If two doughnuts of the same flavor are identical, in how many ways can the treats be distributed if each person refuses to get both of the glazed doughnuts?

28. Circle c_1 has radius 1 and is externally tangent to circle c_2. Both circles are tangent to lines l_1 and l_2, and the two lines intersect at I, which is closer to c_1 than c_2. If $m\angle I = 60^\circ$, then what is the radius of c_2?

29. If $32^x = 9^y$, and $z^x = 243^y$, what is $\sqrt[5]{z^2}$?

30. Nikhil, Bartek, and Eshaan have 3, 3, and 2 colored shirts: Nikhil's are red, Bartek's are white, and Eshaan's shirts are blue. Some of these shirts have flags. Nikhil and Eshaan each have one such shirt and Bartek has two. They want to arrange their shirts so all of each person's shirts are together, and for each flag shirt, there is another flag shirt at most one position away from it. How many ways can they arrange their shirts to satisfy these conditions? The color and presence (or lack) of a flag are the only distinguishing factors of a shirt.

31. BCA is hosting a tournament in which 32 players compete in knockout-round format. The players are all given a ranking between 1 and 32, where 1 is the best and 32 is the worst ranking. However, BCA rigs it so that the tournament plays out in the following way:

- Players 1 and 2 must be in the final round
- Players $1-4$ must be in the semi-finals
- Players $1-8$ must be in the quarter-finals
- Players $1-16$ must be in the octo-finals

If the better-ranked player always wins in a matchup, let N be the number of ways a tournament bracket can be filled out. What is the largest integer k such that 2^k divides N? A tournament is identical if and only if all the matchups are the same.

32. How many ways can Nikhil place 8 disks of different radii on 3 distinct poles so no pole is empty and no disk is above a disk of smaller radius than itself?

33. Clumsy Josh is thinking of a 4-digit number. He claims that when he adds up the four three-digit numbers obtained by crossing out each digit of the original number and reading the other three digits in order, he gets 3308. Julia, however, quickly realizes that Josh made a mistake when he added up the four three-digit numbers. Knowing that Josh's actual sum is less than 3308, what is the largest four-digit number Josh could have been thinking of?

34. Since Jim likes apples so much, his five friends have decided to gift him apples for his upcoming birthday. The first friend gives Jim some positive number of apples, and each subsequent friend, eager to please Jim, gives Jim more apples than he has received thus far. How many ways are there for Jim to receive 45 apples in total?

35. Let ABC be an equilateral triangle with side length 30, inscribed in circle Ω. There exists a square $PQRS$, with PQ on side BC and points R and S on minor arc BC. If the side length of $PQRS$ can be expressed as $w\sqrt{x} - y\sqrt{z}$ for positive integers w, x, y, z, with x and z squarefree, find $w + x + y + z$.

36. Tyrone is a paper boy trying to deliver newspapers in a very strange town. He needs to deliver a paper to house number 169. Unfortunately, his map tells him that he is at the right house, but the house's number only consists of the digits 0, 1, and 2. Tyrone deduces that the house number is in a different base. What is the sum of the possible bases the house number can be in?

37. Nikhil and Jaiden are avid gamblers, and they decide to place bets against each other to perhaps earn some money. On the number line from 1 to 18, Jaiden places an ant on 13. Every second, the ant changes its position by $+1$ with probability $\frac{1}{2}$ and -1 otherwise. If the ant reaches position 1, Jaiden must pay Nikhil 17 dollars, and if the ant reaches position 18, Nikhil must pay Jaiden 17 dollars. The bet is over once the ant reaches either position 1 or 18. What is Jaiden's expected monetary gain, in dollars?

38. Jaiden has recently come into possession of an isosceles right triangle. He randomly picks a point inside the triangle and folds all three vertices of the triangle onto the point. If p is the probability that the 3 folds do not overlap, find the closest integer to $100p$.

39. Two non-zero real numbers, a and b, satisfy $ab = a + b$. Determine the minimum value of

$$4 + \frac{a^2}{b^2} + \frac{b^2}{a^2} + \frac{a}{b} + \frac{b}{a} - ab.$$

40. There are ten people standing equidistant along the perimeter of a circle. Each person randomly chooses someone (possibly themselves!) to "follow". Every second, each person moves to the position that the person they were following was previously at; that is, if person a is at position x at second t, anyone following a would be at position x at second $t+1$. The probability that everyone ends up in their original spot after 120 seconds is equal to $\frac{m}{n}$ where m and n are relatively prime positively integers. What is the sum of the number of positive factors of m and the number of positive factors of n?