# Joe Holbrook Memorial Math Competition

7th Grade

October 16, 2022

1. Since he used all his money after giving 10 dollars to his brother, he had exactly 10 dollars before giving all the money away to his brother.

   Since he used 50 percent of his money to reach 10 dollars, we have $10 \times 2 = 20$.

   Finally, we can multiply 20 by $\frac{5}{4}$ to get $\boxed{25}$ dollars as the starting money for Albert.

2. "Because the nobleman was able to figure out the king's number after the questions, only one number satisfies the king's 3 answers. Let's analyze those possible answers and the numbers that satisfy those answers. (Y indicates yes, N indicates no, the responses are given in the same order as the questions)

   - YYY: 3, 5, 7 satisfy this
   - YYN: 2
   - YNY: 1, 9
   - YNN: 4, 6, 8
   - NYY: 11, 13, ...
   - NYN: no such number satisfies this
   - NNY: 15, 21, ...
   - NNN: 12, 14, ...

   The YYN answer is the only answer that gives one possible number. Hence, this must have been the king's response and $\boxed{2}$ is the king's favorite natural number."

3. Lance has \$240 to start and pays \$115 for the ticket, so he now has $240 - 115 = \boxed{125}$ dollars.

4. The most effective way to cut the cake is to start at a corner and cut each subsequent piece right next to the previous cut. This minimizes the number of leftover pieces of cake that are too small to be turned into a 2" x 2" piece. However, notice that following this method, there will always be a 1" x 4" strip leftover which cannot be turned into any pieces. Therefore, only the 12" x 4" portion is usable, giving us $\frac{12 \cdot 4}{2 \cdot 2} = \boxed{12}$ pieces.

5. We want to find the greatest common factor of 399 and 168. The prime factorization of $399 = 3 \cdot 7 \cdot 19$ and $168 = 2^3 \cdot 3 \cdot 7$. The most 399 and 168 share in their prime factorization is $3 \cdot 7 = 21$. Therefore, $\boxed{21}$ is the greatest common factor of 399 and 168 and so 21 is the greatest number of gift bags that can be made with no items leftover.

6. If I take 5 books, since there are only four subjects, one needs to be repeated. On the other hand, if I only take four books, it is possible that they will all be on different subjects, so the answer is $\boxed{5}$.

7. If $a$ has 9 positive multiples less than 100, then $\frac{100}{a}$ must be equal to 9 with some remainder or 10 with no remainder (since we are only counting multiples LESS than 100, not equal to 100). Using the fact that the quotient decreases as the divisor increases for positive numbers and a constant positive dividend and that $100/9 = 11$ w/remainder, $100/10 = 10$ w/no remainder, $100/11 = 9$ w/remainder, and $100/12 = 8$ w/remainder, $a$ can only be 10 or 11. If $b$ has 19 positive multiples less than 100, then $\frac{100}{a}$ must be equal to 19 with some remainder or 20 with no remainder. Since $100/4 = 25$ w/no remainder, $100/5 = 20$ w/no remainder, and $100/6 = 16$ w/remainder, $b$ can only be 5. Thus, $ab$ is either 50 or 55. However, in either case scenario, there is only 1 multiple of $ab$ strictly less than 100 and that is $ab \cdot 1$. Therefore, $\boxed{1}$ is our answer.

8. Solving the equation, we see that $x = -\frac{k}{3}$, and $-\frac{k}{3} < -\frac{2}{5}$ occurs exactly when $\frac{6}{5} < k$. Thus, the desired probability is $\frac{8\frac{4}{5}}{10} = \frac{22}{25}$ implies $\boxed{47}$.

9. We start by listing a couple of terms of the sequence, suspecting that the sequence might start to repeat. Once we see the first repeat, it is clear that the sequence will be constant from then on. The sequence starts: $4210, 73, 370, 370, \cdots$. We already see two consecutive 370s, so we know the sequence must be constant from the third term on, and thus the answer is $\boxed{370}$.

10. We can narrow down how many non-rotten apples there are by considering the possibilities after reading each statement:

   - 39, 36, 31, 24, 15, or 4
   - 36, 24, 15, or 4
   - $\boxed{15}$

11. First, note that none of the side lengths of the triangle can be 1 foot long because that would not allow a triangle of integer side lengths to be formed (due to the triangle inequality theorem).

   Let the smallest side length of the triangle be 2 feet long. This allows for the other side lengths to be:

   - 3 feet and 4 feet
   - 4 feet and 5 feet
   - 5 feet and 6 feet
   - 6 feet and 7 feet

   Let the smallest side length of the triangle be 3 feet long. This allows for the other side lengths to be:

   - 4 feet and 5 feet
   - 4 feet and 6 feet
   - 5 feet and 6 feet
   - 5 feet and 7 feet

   Let the smallest side length of the triangle be 4 feet long. This allows for the other side lengths to be:

   - 5 feet and 6 feet

   In total, that is $\boxed{9}$ possible triangles Austin can use for the base of his hut.

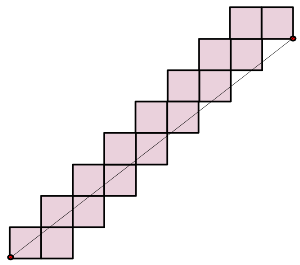
12. This rotation creates a cylinder. The two bases are circles of radius 2, meaning their area would each contribute $4\pi$ to the surface area. The body of the cylinder would have a surface area of the rectangle formed when the cylinder is flattened. This would be the height of the cylinder, which is 4, times the width, which would be the circumference of the bases, which is $4\pi$. Thus, the area would be $16\pi$, and once we add the $8\pi$ from the bases, we get $24\pi$. The integer closest to this is $\boxed{75}$.

13. In order to maximize the median, we need to make the first half of the numbers as small as possible. Since there are 100 people, the median will be the average of the $50th$ and 51st largest amount of cans per person. To minimize the first 49, they would each have one can. Subtracting these 49 cans from the 253 cans gives us 204 cans left to divide among 51 people. Taking 204/51 gives us 4. Therefore, the $50th$ and 51st people could have 4 cans, yielding a median of $\boxed{4}$.

14. We claim that $BA \mid AB$ if and only if $A = B$. If $A = B$, then $AB = BA$, so it follows that $BA \mid AB$. As for the converse, if $BA \mid AB$ and $A \neq B$, then $(AB - BA) \mid AB$, so because $AB = 10A + B$ and $BA = 10B + A$, $9(A - B) \mid AB$. Thus, $9 \mid AB$, so $AB \in \{54, 63, 72, 81\}$. However, for none of these values does $BA$ divide $AB$, so there are $\boxed{9}$ solutions, corresponding to the 9 possible values of $A$ and $B$ (the multiples of 11).

15. Let the bottom red point be $(0, 0)$. Since there are eight layers, the top red point at $(9, 7)$. This means for every one unit right the line goes, it goes up by $\frac{7}{9}$. Since this value is less than 1 but greater than $\frac{1}{2}$, this means the line will usually touch two squares per row in a 2-square zig-zag pattern as seen in the diagram.

However, notice at the portion when the line goes from $\left(4, 3\frac{1}{9}\right)$ to $\left(5, 3\frac{8}{9}\right)$, the line will cross 3 squares on that layer. This is because the line went one unit right, but unlike before, it did not cross into the next layer. Notice that one of the squares is not in the staircase. After this, the line will continue in the 2-square zig-zag pattern because that case does not happen again. Notice that now, only one of the two squares will be in the staircase. Therefore, there will be a total of 4 touched squares which are not in the stair case. Hence, our final answer is $2 \cdot 6 + 3 - 4 = \boxed{11}$.



16. To find the maximum of $y = -\frac{1}{20}x^2 + x + 5$, we need to find the vertex. We know that the x-coordinate of the vertex for a polynomial in the form $ax^2 + bx + c$ is $-b/2a$, so the x-coordinate of the vertex for $y = -\frac{1}{20}x^2 + x + 5$ is $-1/(2 \times -\frac{1}{20}) = 10$. From there, we can plug in 10 for x in the polynomial to find the y-coordinate of the vertex: $y = -\frac{1}{20} \cdot 10^2 + 10 + 5 = 10$. We can do a similar process for the other polynomial and find that the coordinate of the vertex is $(10, 0)$. Therefore, the difference between the maximum and minimum height of the jump rope is $10 - 0 = \boxed{10}$ units.

17. The main idea here is to include a 1 wherever we can in $A$ and $B$, prioritizing higher place values. This is so that we get the greatest value from when it is converted to a 7.

    - For the ten-thousands place, we find that we can have $A = 7\_, \_\_\_$ and $B = 8\_, \_\_\_$.
    - For the thousands place, we can once again only place one 7 to get $A = 77, \_\_\_$ and $B = 83, \_\_\_$.
    - For the hundreds place, we can place a 7 in both, keeping in mind that we have to get a carry over from the tens place; $A = 77, 7\_\_$ and $B = 83, 7\_\_$.
    - For the tens place, we place two 9s to get a carry over for the hundreds place, also keeping in mind that we need a carry over from the ones place; $A = 77, 79\_$ and $B = 83, 79\_$.
    - Finally, we place a 6 and a 9 to get a 5 in the ones place and a carry over for the tens place; our final numbers are $A = 77, 796$ and $B = 83, 799$.

    We confirm that the confused values of $A$ and $B$ are $11, 196$ and $83, 199$, which sum to $94, 395$. Thus, the real values of $A$ and $B$ yield $77, 796 + 83, 799 = \boxed{161, 595}$.

18. Let $a$ be the fraction of the lawn that Alice can mow in an hour, and $b$ be the fraction that Bob can mow in an hour. We know $a + b = \frac{1}{3}$, since they mow the entire lawn in three hours, while $a = \frac{1}{5}$ because Alice takes 5 hours to mow it. So, $b = \frac{1}{3} - \frac{1}{5} = \frac{2}{15}$, and Bob can mow the lawn in $\frac{15}{2}$ hours, or $15 \cdot 30 = \boxed{450}$ minutes.

19. We claim that it is impossible for the product $(x_1 - 1)(x_2 - 2) \cdots (x_n - n)$ to be odd when $n$ is odd. Suppose $n = 2k + 1$. Then, in order for this product to be odd, we must have $k + 1$ even terms and $k$ odd terms in the set $\{x_1, x_2, \cdots, x_n\}$ because in order for the product to be odd, each of the individual terms must be odd as well. But $\{x_1, x_2, \cdots, x_n\} = \{1, 2, \cdots, n\}$, and there are $k$ even terms and $k + 1$ odd terms in the latter set, a contradiction. So, the answer is $\boxed{0}$

20. If $f(10) = 1$, $f(a) = \frac{1}{2} \implies f(a^2) = f(10)$, meaning $a$ must be $\sqrt{10}$. By similar logic, $f(\sqrt[3]{10}) = 1/3$. Thus $f(10^{5/6} = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$, meaning our answer is $10 * 5 * 6 = \boxed{300}$. Notice that this function is the same as the logarithm function of base 10.

21. We know that $N$'s second smallest factor is $p$, its smallest prime factor. Thus, $N$'s second largest factor must be $\frac{N}{p}$. So, we are saying that $\frac{N}{p} = 15p \implies N = 15p^2$. What are the possible values of $p$? Since

$p$ is the smallest prime factor and $N$ is already divisible by 3 and 5, $p$ can only be 2 or 3. So, our final answer is $15 \cdot 2^2 + 15 \cdot 3^2 = \boxed{195}$.

22. The smallest number $n$ such that $s(n) = k$ will have approximately $k/9$ digits, so a larger $k$ means a larger $n$. The smallest integer whose sum of digits is eleven is 29, and the smallest integer whose sum of digits is 29 is $\boxed{2999}$ (we minimize the number of digits by using many 9s, then put the smallest digit in the largest place value.)

23. Every 3 digit number can be expressed in the form $x = 100a+10b+c$. It can be easily seen that the average of all 6 permutations of the number is given by $\frac{100(2a+2b+2c)+10(2a+2b+2c)+(2a+2b+2c)}{6}$, and thus the condition in the problem is that $100a + 10b + c$ is equal to this value, which can be simplified to $37(a + b + c)$. This implies that $37|100a + 10b + c = x$. Thus, 37 necessarily divides into every number that has a nice average, and so the sum $S$ must be divisible by 37, which means that the answer is $\boxed{0}$.

24. The total number of ways to arrange RROOYY is $\frac{6!}{2!2!2!} = 90$, because there are 2 of each letter. Then, we can use PIE to find the total number of ways there will be at least one ROY. Using ROY as a single unit, there are 4! ways to arrange the [ROY], R, O, Y. However, we have counted ROYROY twice, so we subtract one of them to get rid of the overcounting. Thus, the fraction is $\frac{4!-1}{90} = \frac{23}{90}$, so the answer is $\boxed{67}$.

25. This problem is based on Bezout's Lemma, which states that there exists a linear combination of 4 and 7 such that they combine to 1. In essence, the healer needs to give the Hydra enough heads such that it has exactly 1 more than a multiple of 7. We can see that if the healer gives the Hydra 16 heads, this will make it $36 = 7 * 5 + 1$. This will take 4 actions. Finally, the knight will take off 35 of its heads with 5 actions. This is the least amount of actions, as the inverse of 4 modulo 7 is 2, and since the Hydra already has 20 heads, we must get to 28 heads, then 36 heads, and then subtract down. This means our amount of actions must be $4 + 5 = \boxed{9}$

26. Because $O_1O_2 = 6\sqrt{3} < 2(6)$, the circles intersect at two points, which we will call $I_1$ and $I_2$. Denote the midpoint of $\overline{I_1I_2}$ as $M$. Because $O_1M+MO_2 = O_1O_2$, by symmetry $O_1M = MO_2 = 3\sqrt{3}$. We observe that the segments from each center to intersection point is a radius of a circle, so $O_1I_1, O_1I_2, O_2I_1, O_2I_2 = 6$. $3\sqrt{3} = 6/\sqrt{3}$, so $\triangle MO_1I_1, \triangle MO_1I_2, \triangle MO_2I_1$, and $\triangle MO_2I_2$ are congruent $30-60-90$ right triangles. Hence, $I_1M = I_2M = 3$, so the area of each triangle is $(9/2) \cdot \sqrt{3}$. Furthermore, $m\angle I_1O_1I_2 = m\angle I_1O_2I_2 = 60^\circ$, so because the total area of each circle is $36\pi$, the areas of the sectors enclosed by arcs $I_1O_1I_2, I_1O_1I_2$ are both $6\pi$. Again using the symmetry of the desired region, we see that the area of the circles' union can be represented as $2(6\pi - 2[\triangle MO_2I_1])$, which is $2(6\pi - 9\sqrt{3}) = 12\pi - 18\sqrt{3}$. Hence, $a+b+c = 12-18+3 = \boxed{-3}$.

27. There are $\binom{3}{2} = 3$ ways to choose the people who get a glazed doughnut. Assume they are Adisa and Batu (the following logic works for all three choices of people who get glazed donuts). If Adisa's and Batu's other doughnuts are of the same flavor, then there are 2 ways to select which flavor they get, while if they are of different flavors then there are 2 ways to choose which person gets which. (In either case, Catalina's doughnuts are automatically assigned as the ones not given to Adisa and Batu). Hence, there are a total of $3 \cdot (2 + 2) = \boxed{12}$ doughnut distributions.

28. Denote the center of $c_1$ as $O_1$, the center of $c_2$ as $O_2$, the intersection point of $c_1$ and $l_1$ as $P_1$, and the intersection point of $c_2$ as $P_2$. We are trying to find $r$, the radius of $c_2$. Because $m\angle I = 60^\circ$, due to symmetry $m\angle P_2IO_2 = 30^\circ$ and since $l_1$ is tangent to $c_1$ and $c_2$, $m\angle IP_1O_1 = m\angle IP_2O_2 = 90^\circ$. Hence, $\triangle P_1IO_1$ and $\triangle P_2IO_2$ are $30-60-90$ triangles. We know $O_1P_1 = 1$, so $IO_1 = 2$. We also know $IO_2 = 2(O_2P_2) = 2r$, so because $IO_2 = IO_1 + O_1O_2 = IO_1 + 1 + r = 3 + r$, $r = \boxed{3}$.

29. Since $32 = 2^5$ and $9 = 3^2$, $2^{5x} = 3^{2y}$. Since $243 = 3^5$, $z^x = 3^{5y}$. So, $z^x = 2^{x*25/2}$, with $z = 2^{25/2}$. So, $\sqrt[5]{z^2} = \boxed{32}$.

30. If Nikhil's shirts are in the middle, then Nikhil's flag shirt must be within 2 of Eshaan's flag shirt since they both only have one. We see an example of possible flag positions for each person below, left to right Eshaan, Nikhil, Bartek:

$$\_\_F \mid \_F\_\_ \mid FF\_\_\_$$

There are 3 ways to arrange Nikhil and Eshaan's flag shirts and 7 ways to arrange Bartek's shirts. So, in this case, there's $3 \cdot 2 \cdot 7 = 42$ ways to arrange the shirts, since Bartek could be on the left.

If Eshaan's shirts are in the middle, then we get an identical scenario to Nikhil's shirts in the middle, for another 42 cases.

If Bartek's shirts are in the middle, then one flag shirt must be within 2 of one of Eshaan's shirts and Bartek's other flag shirt must be within 2 of one of Nikhil's shirts. Left to right Eshaan, Nikhil, Bartek, we see an example ordering of shirts:

$$\_\_F \mid \_F\_F\_ \mid F\_\_\_$$

There are 3 ways to arrange the flag shirts on Eshaan's side and likewise for Nikhil's side. So, there's a total of $3 \cdot 3 \cdot 2 = 18$ arrangements for this case.

This gives a total of $42 + 42 + 18 = \boxed{102.}$

31. There's only 1 way the final can be played: between Players 1 and 2. From Players 3 and 4, there are 2 ways to arrange the semifinals. There's 4! ways to decide who Players $5-8$ will play in the quarter finals. The same reasoning extends to find $N = 2 \cdot 4! \cdot 8! \cdot 16!$ which gives $\boxed{26}$.

32. Consider placing the disk with largest radii first. There are 3 places it can go. The next-largest disk can go in 3 places as well. This logic continues which gives us $3^8$ ways to place the disks. However, we need to consider the cases in which a pole is empty. If one pole is empty, then using the same logic, there are $2^8 - 2$ ways to place the disks. If two poles are empty, there is 1 way to place the disks. So, the answer is $3^8 - 3 \cdot (2^8 - 2) - 3 = \boxed{5796}$

33. Express Josh's number as $\underline{abcd}$. The four three-digit numbers he obtains by crossing out the individual digits are $\underline{abc}$, $\underline{abd}$, $\underline{acd}$, $\underline{bcd}$. Their sum would therefore be $300a+120b+21c+3d$. Notice, that no matter what the values of a, b, c, and d are, the sum must be divisible by 3. Using this information, Julia quickly understood that Josh had made a mistake, as 3308 is not divisible by 3. Since 3306 is the largest number divisible by 3 less than 3308, it would be the optimal sum, as the larger the number, the larger its sum. Thus, the largest number Josh could be thinking of, abcd, must satisfy $3306 = 300a + 120b + 21c + 3d$. Now, to maximize $\underline{abcd}$, we want to maximize each digit, in order from $a$ to $d$. We can first set $a = 9$, leaving $120b + 21c + 3d = 606$. So, $b = 5$. Using this process, we find that $\boxed{9502}$ is the largest such number.

34. Suppose the first friend gives Jim $a$ apples. Then, each subsequent friend must give Jim $a + b$, $2a + b + c$, $4a + 2b + c + d$, and $8a + 4b + 2c + d + e$ apples, for positive integers $a, b, c, d, e$, such that each friend gives Jim more apples than he has received so far. So, the total number of apples he receives is $16a + 8b + 4c + 2d + e = 45$. Since all 5 variables are positive integers, if we let $a' = a - 1$, and similarly for the other 4 variables, we have $16a' + 8b' + 4c' + 2d' + e' = 14$.

Clearly, $a'$ must be 0. If $b' = 1$, then we have $4c' + 2d' + e' = 6$. If $c' = 1$, then we have 2 solutions, and if $c' = 0$, then we have 4 solutions. If $b' = 0$, then we have $4c' + 2d' + e' = 14$. If $c' = 3, 2, 1, 0$, then we have 2, 4, 6, and 8 solutions respectively.

In total, we have $2 + 4 + 2 + 4 + 6 + 8 = \boxed{26}$.

35. Let the side length of $PQRS$ be $2s$. Then, if we let $O$ be the center of circle $\Omega$ and $M$ be the midpoint of $PQ$, we have that $MO = \dfrac{BC/2}{\sqrt{3}} = 5\sqrt{3}$. Now, let $N$ is the midpoint of $RS$. Since $OS = OB = 2 \cdot OM = 10\sqrt{3}$, $ON = OM + MN = 5\sqrt{3} + 2s$ and $SN = s$, applying the Pythagorean Theorem to right triangle $OSN$ gives us

$$s^2 + (2s + 5\sqrt{3})^2 = 300.$$

If we expand and simplify, we obtain $5s^2 + 20\sqrt{3}s - 225 = 0$, which we can simplify further to obtain $s^2 + 4\sqrt{3}s - 75 = 0$. Finally, applying the quadratic formula gives us $s = \sqrt{57} - 2\sqrt{3}$, and so our sidelength is $2s = 2\sqrt{57} - 4\sqrt{3}$. The answer is $\boxed{66}$.

36. We know that the maximum possible base we need to check is base 169, since the number 169 is simply itself if it is converted to any base greater than 169. We can limit the possible bases by looking at what bases lead to the last digit of 169 after it has converted to be 0, 1, or 2. This entails looking at what numbers leave a remainder of 0,1,or 2 when they divide 169. This can also be found by finding the numbers that divide 169, 168, and 167. So from this we can find that if 169 is converted into the following bases, the units digit of the converted number will be 0, 1, or 2: $2, 3, 4, 6, 7, 8, 12, 13, 14, 21, 24, 42, 56, 84, 167, 168, 169$. We immediately know that base 2 and 3 are possible since all the digits in base 2 and 3 are 0, 1, or 2. We also know base 13 and 169 work since 169 base 13 is 100 and 169 base 169 is 10. We know that 14, 21, 24, 42, 56 will all not work as bases since they leave a remainder larger than 2 when they divide

169. We can try converting 169 into base 4, 6, 7, 8, and 12 and see that only base 4 and base 12 work. Finally, we can test that 84, 167, and 168 work. Therefore, the possible bases of the house numbers are: $2, 3, 4, 12, 13, 84, 84, 167, 168, 169$ which sum to $\boxed{622}$.

37. If we let $a_n$ be the expected monetary gain for Jaiden when the ant is at integer $n$, we have that $a_n = \frac{1}{2}a_{n-1} + \frac{1}{2}a_{n+1}$ with $a_1 = -17, a_{18} = 17$. However, note that the recurrence relation represents one of an arithmetic sequence, meaning that $a_1, a_2, \ldots$ is an arithmetic sequence. The common difference is $\frac{a_{18} - a_1}{17} = \frac{34}{17} = 2$, so $a_{13} = a_1 + 12 \times 2 = -17 + 24 = \boxed{7}$.

38. Let the isosceles triangle be $ABC$ with a right angle at $\angle B$, and let the point chosen be $P$. The three creases that Jaiden folds along will be the perpendicular bisectors of line segments $PA$, $PB$, $PC$. As long as these three perpendicular bisectors do not intersect inside the triangle, the folds will not overlap. Consider the perpendicular bisectors of $\overline{PA}$ and $\overline{PB}$. These will meet at the circumcenter of triangle $PAB$ (since the intersection point will be equidistant from $P$, $A$, and $B$). This circumcenter will be outside $\triangle ABC$, or equivalently, on the other side of $\overline{AB}$ as point $P$, if and only if $m\angle APB > 90$ degrees.

The set of points $P$ such that $m\angle APB > 90$ degrees is the set of point inside the semicircle with diameter $AB$. Equivalently, if we construct the semicircle with diameter $BC$ and $CA$, we need $P$ to lie within all three circles. We can split this region into two by drawing a line from $B$ to the midpoint of $\overline{AC}$. Each of these two regions is a quarter-circle minus a smaller isosceles right triangle. If we let the $AB = 2$, the area of each of these regions is $\frac{\pi}{4} - \frac{1}{2}$, and so our total area is $\frac{\pi}{2} - 1$. Since the total area of the triangle is 2, our answer is $\frac{\frac{\pi}{2} - 1}{2} \approx 0.29 \implies \boxed{29}$.

39. Starting with only one equation, we try to manipulate it to resemble the expression we want to minimize. We notice that dividing the given equation by $a$ and by $b$ gives us $b = 1 + \frac{b}{a}$ and $a = 1 + \frac{a}{b}$, respectively. Multiplying these two equations gives us

$$ab = 2 + \frac{a}{b} + \frac{b}{a}.$$

Substituting this expression of $ab$ into the expression we want to minimize gives us

$$\begin{aligned} &4 + \frac{a^2}{b^2} + \frac{b^2}{a^2} + \frac{a}{b} + \frac{b}{a} - \left(2 + \frac{a}{b} + \frac{b}{a}\right) \\ =&2 + \frac{a^2}{b^2} + \frac{b^2}{a^2}. \end{aligned}$$

Since $\frac{a^2}{b^2}$ and $\frac{b^2}{a^2}$ are both positive, we can use AM-GM to get

$$\frac{\frac{a^2}{b^2} + \frac{b^2}{a^2}}{2} \geq \sqrt{\frac{a^2}{b^2} \cdot \frac{b^2}{a^2}} = 1$$

with equality when $\frac{a^2}{b^2} = \frac{b^2}{a^2}$. This can be obtained when $a = b$. We see $a = b = 2$ is a solution to the given $ab = a + b$ so the minimum value of $\frac{a^2}{b^2} + \frac{b^2}{a^2}$ is 2. Hence, our answer is $2 + 2 = \boxed{4}$.

40. First, we note that there are $10^{10}$ total configurations: each of the 10 people can choose one of 10 people to follow. Now, note that whenever two people follow the same person, they can never both return to their original spots after 120 seconds. Thus, our possible configurations narrow down to only permutations. We note that the only cases where each person does not return to their original position is if there is a permutation cycle of length 7 or 9; all other natural numbers $\leq 10$ are factors of 120. The number of possible 7-cycles is $\binom{10}{3} \times 3! \times 6!$; the first two terms represent the process of choosing and permuting the people not in the 7-cycle, and the last term represents the number of ways to create a cycle from 7 people. In a similar fashion, the number of possible 9-cycles is $\binom{10}{1} \times 1! \times 8!$. So the total number of valid permutations is equal to $10! - \binom{10}{3} \times 3! \times 6! - \binom{10}{1} \times 1! \times 8! = 10! \times (1 - \frac{1}{7} - \frac{1}{9}) = 10! \times \frac{47}{63}$.

Thus, the probability that everyone ends up in their original spot is equal to $\frac{10! \times \frac{47}{63}}{10^{10}}$, which, after some simplifying, is equal to $\frac{3^2 \times 47}{2^2 \times 5^8} \rightarrow 3 \times 2 + 3 \times 9 = \boxed{33}$.