Joe Holbrook Memorial Math Competition

8th Grade Solutions

October 16, 2022

1. Based on the percentages given, we can write $\frac{24}{100}x = \frac{30}{100}y$. Therefore, we find that $\frac{x}{y} = \frac{24}{30}$, which is equivalent to $\boxed{80}$%.

2. We can simply try each prime - we find that $3^2 + 4$, $5^2 + 4$, and $7^2 + 4$ are prime but $\boxed{11}^2 + 4 = 125$ is not.

3. Assume Tyrone's machine can make t snowballs in one hour, Pablo's machine can make p snowballs in one hour, and Uniqua's machine can make u snowballs in one hour. We are given. $t + u = 32$, $u + p = 45$, and $p + u + t = 57$. We want to find what $t + p$ is equal to. Notice that $(t + u) + (u + p) + (t + p) = 2(p + u + t)$. This equation can be rearranged so we know that $t + p = 2(p + u + t) - (u + p) - (t + u) = 2(57) - 45 - 32 = 37$. Therefore, Tyrone and Pablo's machine can make $\boxed{37}$ snowballs in one hour.

4. It takes $\frac{450}{60} \cdot 60 = 450$ minutes to drive (distance is speed times time, and there are 60 minutes in an hour). Similarly, it takes $\frac{400}{240} \cdot 60 + 120 = 220$ minutes to fly (the extra 120 minutes is from the two hours of arriving/boarding), and the answer is therefore $450 - 220 = \boxed{230}$.

5. Let S, C, and P represent the number of snakes, chicken, and parrots respectively. Since there are 24 total animals, it is known that $S + C + P = 24$. However, because only chickens and parrots have legs, $38 = 2C + 2P$, which means that, in total, there are 19 chickens and parrots combined. Therefore, there must be $\boxed{5}$ snakes.

6. Let Uniqua's flying speed be u mph and the wind speed be w mph. Since distance is equal to speed multiplied by time, when Uniqua flew north for an hour, she traveled $u - w$ miles. When she flew south for two hours, she traveled $2(u+w)$ miles. This also means she is $2(u+w) - (u-w) = u + 3w = 1200$ miles from where she started. If it takes Uniqua 3 hours to fly back to where she started, that means $\frac{1200}{u - w} = 3$. Now we have two equations which we can use to solve for u and w. $1200 = 3u - 3w \implies 400 = u - w$. Subtracting this equation from $u + 3w = 1200$ gives us that $4w = 800$ or $w = \boxed{200}$ miles per hour.

7. We can narrow down how many non-rotten apples there are by considering the possibilities after reading each statement:

   - The first statement tells us that the number of non-rotten apples will be 40 minus some perfect square. This gives the values of 39, 36, 31, 24, 15, or 4.
   - Out of the numbers given above, we can only have the ones that are divisible by either 2 or 5. These are 36, 24, 15, or 4.
   - The only odd number in the list is $\boxed{15}$.

8. We note that for any fishy number n, n cannot be a one digit number: if so, then because its only digit would be prime and n itself would be prime, contradicting the problem. Therefore, $n \geq 10$. Both of n's digits must be elements of $\{0, 4, 6, 8, 9\}$. Since $n > 2$, n cannot be even as this would make n composite, so the units' digit of n must be 9, making the only possible fishy numbers $49, 69, 89$, and 99. Because $49 = 7^2, 69 = 3 * 23$, and $99 = 3^2 * 11$, the prime number 89 is the only fishy number, so the sum is $\boxed{89}$.

9. Using the distance formula, $AB = 10$. Jack thus travels a distance of 10, while Jane travels a distance of $\frac{10\pi}{2} = 5\pi$. Thus, the difference is $5\pi - 10$, and the answer is $\boxed{15}$.

10. The enclosed area is the same as an equilateral triangle minus the three sectors of the circles, each of which have an interior angle of 60°. The area of the triangle is $4\sqrt{3} = \sqrt{48}$ and the area of the sectors is equal to $\frac{1}{6} \times 3 \times 4\pi = 2\pi$, so the total enclosed area is $\sqrt{48} - 2\pi \to \boxed{50}$.

11. Since the area of the triangle plus the area of the pentagon is just the area of the entire square (which is $30^2 = 900$), the triangle has area $\frac{900}{3} = 300$. The area of the triangle is also $\frac{1}{2} \cdot AE \cdot AF = \frac{1}{2} \cdot AE^2$ by area formulas, so $AE^2 = 300 \cdot 2 = \boxed{600}$.

12. The smallest number n such that $s(n) = k$ will have approximately $k/9$ digits, so a larger k means a larger n. The smallest integer whose sum of digits is eleven is 29, and the smallest integer whose sum of digits is 29 is $\boxed{2999}$ (we minimize the number of digits by using many 9s, then put the smallest digit in the largest place value.)

13. The main idea here is to include a 1 wherever we can in A and B, prioritizing higher place values. This is so that we get the greatest value from when it is converted to a 7.

    - For the ten-thousands place, we find that we can have $A = 7\_, \_\_\_$ and $B = 8\_, \_\_\_$.
    - For the thousands place, we can once again only place one 7 to get $A = 77, \_\_\_$ and $B = 83, \_\_\_$.
    - For the hundreds place, we can place a 7 in both, keeping in mind that we have to get a carry over from the tens place; $A = 77, 7\_\_$ and $B = 83, 7\_\_$.
    - For the tens place, we place two 9s to get a carry over for the hundreds place, also keeping in mind that we need a carry over from the ones place; $A = 77, 79\_$ and $B = 83, 79\_$.
    - Finally, we place a 6 and a 9 to get a 5 in the ones place and a carry over for the tens place; our final numbers are $A = 77, 796$ and $B = 83, 799$.

    We confirm that the confused values of A and B are $11, 196$ and $83, 199$, which sum to $94, 395$. Thus, the real values of A and B yield $77, 796 + 83, 799 = \boxed{161, 595}$.

14. Call the length of one diagonal $2a$ and the other $2b$. The perimeter given as 40 tells us that each side must be 10. Since diagonals of a rhombus intersect at right angles and bisect one another, we have a right triangle of legs with side lengths a and b, and by Pythagorean Theorem we have $a^2 + b^2 = 100$.

    By the formula for the area of a rhombus, we also have that $2ab = 96$. Adding the two equations and taking the positive square root gives us $a + b = 14$, so the sum of the diagonals is $\boxed{28}$.

15. We know that N's second smallest factor is p, its smallest prime factor. Thus, N's second largest factor must be $\frac{N}{p}$. So, we are saying that $\frac{N}{p} = 15p \implies N = 15p^2$. What are the possible values of p? Since p is the smallest prime factor and N is already divisible by 3 and 5, p can only be 2 or 3. So, our final answer is $15 \cdot 2^2 + 15 \cdot 3^2 = \boxed{195}$.

16. Every 3 digit number can be expressed in the form $x = 100a + 10b + c$. Then, the permutations of the number $\overline{abc}$ would be $100a + 10b + c$, $100a + 10c + b$, $100b + 10a + c$, $100b + 10c + a$, $100c + 10a + b$, and $100c + 10b + a$. Therefore, the average of all 6 permutations of the number is given by

$$\frac{100(2a + 2b + 2c) + 10(2a + 2b + 2c) + (2a + 2b + 2c)}{6},$$

    and this can be simplified to $37(a + b + c)$. Since the condition in the problem is that $100a + 10b + c$ is equal to this value, we must have that $37 | 100a + 10b + c = x$. Thus, 37 necessarily divides into every number that has a nice average, and so the sum S must be divisible by 37. Therefore, the remainder is $\boxed{0}$.

17. Let O be the center of the semicircle. By symmetry, $OC = OD = \frac{6}{2} = 3$, and by the Pythagorean Theorem, the radius OB equals $\sqrt{7^2 + 3^2} = \sqrt{58}$. The area of the semicircle is therefore $\frac{1}{2}\left(\sqrt{58}\right)^2 \pi = 29\pi$, so $a = \boxed{29}$.

18. The probability can be expressed as the ratio of areas of the region. We can also find the area of the complement, since when the diagram is drawn out, it ends up being a triangle with vertices $\left(\frac{3}{2}, \frac{1}{2}\right)$, $(6, 2)$ and $(0, 2)$. The area of this is just $6 \cdot \frac{3}{2} \cdot \frac{1}{2} = \frac{9}{2}$. Then, the total area of the trapezoid is $\frac{1}{2}(6 + 2) \cdot 2 = 8$. This means that the area we are looking for is $8 - \frac{9}{2} = \frac{7}{2}$, and the ratio between areas is $\frac{\frac{7}{2}}{8} = \frac{7}{16}$, giving an answer of $\boxed{23}$.

19. The number of Mayan digits needed to write a base-10 integer k is equal to the least power of 20 that is greater than k, so we want the minimum n such that $20^n > 10^{20}$.

    Since $20 = 10 \cdot 2$, we can rewrite the expression as $(10 \cdot 2)^n > 10^{20}$ or $2^n > 10^{20-n}$. We observe that $2^{10} > 10^3$, so if $\frac{n}{20-n} \geq \frac{10}{3}$, then the desired inequality will be true. This occurs when $n \geq \left\lceil \frac{200}{13} \right\rceil = 16$, so any $n \geq 16$ will be sufficient.

    Meanwhile, because $65536 = 2^{16} < 10^5$, we can establish a lower bound for n: if $\frac{n}{20-n} \leq \frac{16}{5}$ (which occurs when $n \leq \left\lceil \frac{320}{21} \right\rceil = 15$), then $2^n < 10^{20-n}$. Thus, to satisfy both inequalities, n must be greater than 15, and it can be any integer that is at least 16. Hence, the least number of Maya digits is $\boxed{16}$.

20. If a team wins against another team, this means the sum of all the points for all teams increases by 3; if they tie, the sum increases by 2. So the maximum sum is achieved when there are no ties, whereas the minimum sum is achieved when there are only ties. So $M - N = \binom{6}{2} \times 2 \times (3-2) = \boxed{30}$

    A team winning will mean the two teams collectively gain 3 points instead of 2, so $M - N$ is the number of games played: $6 \cdot 5 = \boxed{30}$.

21. We can rewrite the expression as

$$\frac{\lfloor \sqrt[5]{1} \rfloor}{\lfloor \sqrt[5]{2} \rfloor} \cdot \frac{\lfloor \sqrt[5]{3} \rfloor}{\lfloor \sqrt[5]{4} \rfloor} \cdot \frac{\lfloor \sqrt[5]{5} \rfloor}{\lfloor \sqrt[5]{6} \rfloor} \cdots \frac{\lfloor \sqrt[5]{2021} \rfloor}{\lfloor \sqrt[5]{2022} \rfloor}.$$

    Note that each of these terms will cancel out except for the even perfect 5th powers, at which the denominator will be greater than 1. At odd perfect 5th powers, the denominator will be above the floor and thus it will still cancel. Thus, we can write the expression as

$$\frac{\lfloor \sqrt[5]{31} \rfloor}{\lfloor \sqrt[5]{32} \rfloor} \cdot \frac{\lfloor \sqrt[5]{1023} \rfloor}{\lfloor \sqrt[4]{1024} \rfloor} = \frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8}.$$

    Thus, the answer will be $3 + 8 = \boxed{11}$.

22. Because of the condition that Amir cannot walk through points that have an odd x and y coordinate, he can only go to points (a, b) where at least one of $\{a, b\}$ is even. In order to do so, he must take two consecutive steps in a given direction in order to be able to move perpendicularly.

    Hence, we can consider his path as a sequence of 8 moves: 4 that consist of 2 consecutive steps right and 4 that consist of 2 consecutive steps up. These 8 moves can be arranged in any way to form a unique path: because the 4 right-steps are identical, and so are the 4 up-steps, the number of possible sequences is

$$\frac{8!}{4! \cdot 4!} = \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2 \cdot 1} = \boxed{70}.$$

23. The dime's center can land in the concentric circle of radius 5, sharing the same center as the one with radius 6. This gives a total area of 25π in which it can land in. Then, for it to land within the equilateral triangle as well, the dime's center must lie within another equilateral with side length $4\sqrt{3}$; this new triangle's center is the same as the original circle's center. The area in which the dime's center can be in is $\frac{(4\sqrt{3})^2\sqrt{3}}{4} = 12\sqrt{3}$. Thus, the probability is $\frac{12\sqrt{3}}{25\pi}$, so the answer would be $12 + 3 + 25 = \boxed{40}$.

24. Since the boulder is tangent to the mountain and the ground, the radii of the boulder perpendicular to the ground and mountain section off a square with a side length of 5 feet. The diagonal of the square is $5\sqrt{2}$ feet long and passes through the center of Tasha's rock. That means we know that $5\sqrt{2} = 5 + r +$ the distance from the center of Tasha's rock to where the ground and mountain meet. Notice that the radii of Tasha's rock perpendicular to the ground and mountain section off a square with diagonal $r\sqrt{2}$. This means $5\sqrt{2} = 5 + r + r\sqrt{2} \implies r = \frac{5(\sqrt{2}-1)}{\sqrt{2}+1} = 5(3 - 2\sqrt{2}) = 15 - 10\sqrt{2}$. Therefore, the answer to this problem is $(15 - 10\sqrt{2}) + (15 + 10\sqrt{2}) = \boxed{30}$.

25. There are $\binom{3}{2} = 3$ ways to choose the people who get a glazed doughnut, since no person can get both of them. Assume they are Adisa and Batu (the following logic works for all three choices of people who get glazed donuts). If Adisa's and Batu's other doughnuts are of the same flavor, then there are 2 ways

to select which flavor they get, while if they are of different flavors then there are 2 ways to choose which person gets which. In either case, Catalina's doughnuts are automatically assigned as the ones not given to Adisa and Batu. Hence, there are a total of $3 \cdot (2+2) = \boxed{12}$ doughnut distributions.

26. There's only 1 way the final can be played: between Players 1 and 2. From Players 3 and 4, there are 2 ways to arrange the semifinals. There's 4! ways to decide who Players $5-8$ will play in the quarter finals. The same reasoning extends to find $N = 2 \cdot 4! \cdot 8! \cdot 16!$ which gives $\boxed{26}$.

27. Consider placing the disk with largest radii first. There are 3 places it can go. The next-largest disk can go in 3 places as well. This logic continues which gives us 3^8 ways to place the disks. However, we need to consider the cases in which a pole is empty. If one pole is empty, then using the same logic, there are $2^8 - 2$ ways to place the disks. If two poles are empty, there is 1 way to place the disks. So, the answer is $3^8 - 3 \cdot (2^8 - 2) - 3 = \boxed{5796}$.

28. Pollo and Chicken each can roam in a quarter circle of radius 4. Label Pollo's vertex of the pen as A and Chicken's vertex of the pen as B. Call the two intersection points of these quarter circles as P and Q.

    We know that the hypotenuse of the triangle of sides 4 and $4\sqrt{2}$ is $4\sqrt{3}$. Half of this will give the height of each of the two identical triangles constructed from AQP and BQP. These will have height $2\sqrt{3}$, and since we know the radius to be 4, the base will be 2 by the Pythagorean theorem. The ratio between these side lengths tell us that this is a $30-60-90$ triangle.

    Thus, the sector of 60° of a circle of radius 4 is $\frac{4^2\pi}{6} = \frac{8\pi}{3}$. Subtracting out the equilateral triangle, we get half the region in which their roaming areas overlap. This will be $\frac{8\pi}{3} - 4\sqrt{3}$, and multiplying by 2, we get $\frac{16\pi}{3} - 8\sqrt{3}$.

    Since the area in which they can each individually roam is $\frac{4^2\pi}{4} = 4\pi$, we can divide their overlapping area over this value and get $\frac{4}{3} - \frac{2\sqrt{3}}{\pi}$. Thus our answer is $12 + 4 + 3 = \boxed{19}$.

29. Express Josh's number as $\underline{abcd}$. The four three-digit numbers he obtains by crossing out the individual digits are $\underline{abc}$, $\underline{abd}$, $\underline{acd}$, $\underline{bcd}$. Their sum would therefore be $300a+120b+21c+3d$. Notice, that no matter what the values of a, b, c, and d are, the sum must be divisible by 3. Using this information, Julia quickly understood that Josh had made a mistake, as 3308 is not divisible by 3. Since 3306 is the largest number divisible by 3 less than 3308, it would be the optimal sum, as the larger the number, the larger its sum. Thus, the largest number Josh could be thinking of, $\underline{abcd}$, must satisfy $3306 = 300a + 120b + 21c + 3d$. Now, to maximize $\underline{abcd}$, we want to maximize each digit, in order from a to d. We can first set $a = 9$, leaving $120b + 21c + 3d = 606$. So, $b = 5$. Using this process, we find that $\boxed{9602}$ is the largest such number.

30. The prime factorization of 2023 is $7 \cdot 17^2$, which can be found through applying the divisibility rule by 7 or trial division. Thus, if $2023 \mid n^2$, then n^2 is divisible by both 7 and 17^2, so n is divisible by 7 and 17. However, because $2023 \nmid n$, n cannot be divisible by both 7 and 17^2. Hence, $7 \mid n, 17 \mid n$, and $17^2 \nmid n$, so n is a multiple of 119 and so is $n - 2023k$ for any k. Thus, all remainders will be divisible by 119. There are no other restrictions on n, so because 17 is prime, the remainder can be any positive multiple of 119 less than 2023: $119(1), 119(2), \ldots, 119(16)$. These integers have sum $\dfrac{119 \cdot 16 \cdot 17}{2} = 2023 \cdot 8 = \boxed{16184}$.

    (Note that because $2023 \mid n$ corresponds to n having a remainder of 0 when divided by 2023, the condition that n is not divisible by 2023 does not change the sum of remainders.)

31. In order to satisfy the condition that each of Jim's friends gets at least one apple, assume they each start with an apple. Therefore, we are now trying the find the number of ways Jim can distribute 9 apples amongst his 4 friends. Using stars and bars, there are $\dbinom{9+4-1}{4-1} = \dbinom{12}{3} = 220$ ways to distribute the apples without any further restrictions. From here, we can subtract the cases where each of Jim's friends gets a different number of apples. This can only happen if the apple distribution is: $(1,2,3,7), (1,2,4,6), (1,3,4,5)$, and there are 4! ways each of these cases can happen. Therefore, the solution is $220 - 3(4!) = \boxed{148}$.

32. Notice that we can complete the square inside each of these absolute value signs. This becomes

$$|(x+1)^2 + 5| + \left|\left(y - \frac{7}{2}\right)^2 + \frac{7}{4}\right| = 13.$$

Since squared terms are guaranteed to be positive, and what we're adding on each of them is also positive, we can just drop the absolute value signs. Now, moving the constants onto the right hand side, we get the equation $(x+1)^2 + \left(y - \frac{7}{2}\right)^2 = \left(\frac{5}{2}\right)^2$.

Thus, the area of this will be $\frac{5^2\pi}{2^2}$, and this is closest to $\boxed{20}$.

33. In base 6, these integers range from 1 to 5555. If we add leading zeros such that they each have four digits, we can basically consider these numbers as strings of four digits, each ranging from 0 to 5, but 0000 doesn't count. Also, adding 0s naturally does not change the number of non-zero digits. If there are two non-zero digits, we have $\binom{4}{2} \cdot 5^2 = 150$ choices ($\binom{4}{2}$ is choosing the positions for the non-zero digits, and 5^2 is choosing what they are). Similarly, there are $\binom{4}{4} \cdot 5^4 = 625$ integers with four non-zero digits, so the total is $150 + 625 = \boxed{775}$.

34. We can split our desired sum into $N = \sqrt{r} + \sqrt{s}$ and $K = \frac{1}{r} + \frac{1}{s}$. Let us first focus on $N = \sqrt{r} + \sqrt{s}$. If we square both sides of this equation, we get $N^2 = r + 2\sqrt{rs} + s$. By Vieta's Formulas, we know that the sum of the solutions to the quadratic is $-\left(\frac{-35}{1}\right) = 35$ and their product is $\frac{49}{1} = 49$. Hence, $r + s = 35$, and $rs = 49$. Plugging this into our equation, we get $N^2 = 35 + 2\sqrt{49} = 35 + 2 \cdot 7 = 49$, so $N = \pm 7$. However, we are told that r and s are positive real numbers, so $N = \sqrt{r} + \sqrt{s}$ must be positive, meaning that $N = 7$.

   Now, we can calculate $K = \frac{1}{r} + \frac{1}{s}$. This expression can be rewritten with a common denominator of rs as

$$K = \frac{s}{rs} + \frac{r}{rs} = \frac{r+s}{rs} = \frac{35}{49} = \frac{5}{7}.$$

   Finally, we can add the values of N and K together: $N + K = 7 + \frac{5}{7} = \frac{49}{7} + \frac{5}{7} = \frac{54}{7}$. Since 54 and 7 share no common factors greater than 1, $\frac{m}{n} = \frac{54}{7}$, and therefore, $m + n = 54 + 7 = \boxed{61}$.

35. Consider quadrilateral $ABDE$. Since $AB = DE = 10$ and $AB \parallel DE$, $ABDE$ must be a parallelogram. Then, $BD = AE = 10$. So, since $BD = BC = CD = 10$, $\triangle BCD$ is an equilateral triangle. The area of $ABDE$ is then the sum of the areas of $\triangle BCD$, which is $\frac{10^2\sqrt{3}}{4} = 25\sqrt{3}$, and the area of $ABDE$. The area of $ABDE$ is clearly maximized when it is a square (since the height is largest then), which has an area of 100. Now, for the minimum, $[ABDE]$ can theoretically get arbitrarily small, but given that $ABCDE$ is convex, the most "bent" $ABDE$ can be is if two of its angles are 120° (then, one of the angles in $ABCDE$ is 180°). In this case, the height of $ABDE$ is $5\sqrt{3}$, and so its area is $50\sqrt{3}$.

   So, finally, $m = 50\sqrt{3} + 25\sqrt{3} = 75\sqrt{3}$ and $n = 100 + 25\sqrt{3}$. Then, $m + n = 100 + 100\sqrt{3} \implies \boxed{203}$.

36. We will use complimentary counting. Since one of A or $S - A$ must contain 1, that subset will have a gcd of 1. Therefore, the only way for $\gcd(A) \neq \gcd(S - A)$ is if one of A or $S - A$ contains numbers that are all have a common factor (that isn't 1). We will assume for now that is it A, and multiply by 2 in the end. If all elements of A are divisible by 2, then its elements must come from $\{2, 4, 6, 8, 10\}$. Remembering to not include the empty set, that's $2^5 - 1$ options. If all elements of A are divisible by 3, then its elements come from $\{3, 6, 9\}$. We have to take out the empty set, but also $\{6\}$, since this was already counted in the first case. That's $2^3 - 2 = 6$ options. If all elements are divisible by 4, then we've already counted them in the first case. If all elements are divisible by 5, then the elements are from $\{5, 10\}$. We have to take out the empty set and $\{10\}$, since it was already counted. This gives $2^2 - 2 = 2$ options. Finally, the only case we haven't yet counted is if all the elements are divisible by 7, which is just 1 possibility ($\{7\}$).

   In total, there are $2(31 + 6 + 2 + 1) = 80$ ways for $\gcd(A) \neq \gcd(S - A)$, and so our final answer is $2^{10} - 80 = \boxed{944}$.

37. Let the isosceles triangle be ABC with a right angle at $\angle B$, and let the point chosen be P. The three creases that Jaiden folds along will be the perpendicular bisectors of line segments PA, PB, PC. As long as these three perpendicular bisectors do not intersect inside the triangle, the folds will not overlap. Consider the perpendicular bisectors of $\overline{PA}$ and $\overline{PB}$. These will meet at the circumcenter of triangle PAB (since the intersection point will be equidistant from P, A, and B). This circumcenter will be outside $\triangle ABC$, or equivalently, on the other side of $\overline{AB}$ as point P, if and only if $m\angle APB > 90^\circ$.

   The set of points P such that $m\angle APB > 90^\circ$ is the set of point inside the semicircle with diameter

AB. Equivalently, if we construct the semicircle with diameter BC and CA, we need P to lie within all three circles. We can split this region into two by drawing a line from B to the midpoint of $\overline{AC}$. Each of these two regions is a quarter-circle minus a smaller isosceles right triangle. If we let the $AB = 2$, the area of each of these regions is $\frac{\pi}{4} - \frac{1}{2}$, and so our total area is $\frac{\pi}{2} - 1$. Since the total area of the triangle is 2, our answer is $\frac{\frac{\pi}{2} - 1}{2} \approx 0.29 \implies \boxed{29}$.

38. Consider first the primes that divide 2023: 7, 17. We can see from the $7a$ term that the expression will have at least one power of 7. Similarly, we can see since $\gcd(17b, \text{lcm}(2023, c))$ has at least one factor of 17, so will the desired expression. This means the only possible values of the expression are $1071, 1190, 1309, 1428$.

    We see that $7a$ can have a maximum value of $17 \cdot 12 = 204$, since adding any other factor will bring the expression out of the bounds. We can also see that there are no other limits on b, c so we can maximize them however we wish: setting them to $1499, 1498$ will maximize $a+b+c$ while still satisfying the bounds. Thus, our (a, b, c) is $204, 1499, 1498$ which sums to $\boxed{3201}$.

39. Let S be the number of paths that start and end in different directions. Ending in different directions means that there's an odd number of turns, so in any valid path there's an even number of right turns and an odd number of left turns. This means any valid path in S can be reflected over the diagonal $y = x$ to create an invalid path in S. So, our answer is $\frac{S}{2}$. To calculate S, consider a first move of up and a last move of right. This is equivalent to traveling from (0, 1) to (9, 10) with no restrictions. Thus $S = 2 \cdot \binom{18}{9}$ and our answer is $\binom{18}{9} = \boxed{48620}$.

40. First, we note that there are 10^{10} total configurations: each of the 10 people can choose one of 10 people to follow. Now, note that whenever two people follow the same person, they can never both return to their original spots after 120 seconds. Thus, our possible configurations narrow down to only permutations.

    We note that the only cases where each person does not return to their original position is if there is a permutation cycle of length 7 or 9; all other natural numbers ≤ 10 are factors of 120. The number of possible 7-cycles is $\binom{10}{3} \times 3! \times 6!$; the first two terms represent the process of choosing and permuting the people not in the 7-cycle, and the last term represents the number of ways to create a cycle from 7 people.

    In a similar fashion, the number of possible 9-cycles is $\binom{10}{1} \times 1! \times 8!$. So the total number of valid permutations is equal to $10! - \binom{10}{3} \times 3! \times 6! - \binom{10}{1} \times 1! \times 8! = 10! \times (1 - \frac{1}{7} - \frac{1}{9}) = 10! \times \frac{47}{63}$. Thus, the probability that everyone ends up in their original spot is equal to $\frac{10! \times \frac{47}{63}}{10^{10}}$, which, after some simplifying, is equal to $\frac{3^2 \times 47}{2^2 \times 5^8} \to 3 \times 2 + 3 \times 9 = \boxed{33}$.