Joe Holbrook Memorial Invitational Competition

4th Grade Solutions

March 19, 2023

1. Compute $\left(2+\frac{9}{2+1}\right)\times\left(8+\frac{63}{8+1}\right)$.

   Answer: $\boxed{75}$

   Solution: We compute to get

$$\left(2+\frac{9}{3}\right)\times\left(8+\frac{63}{9}\right)=5\cdot 15=\boxed{75}$$

2. There is an equilateral triangle with side length 12 and a square with side length 9. What is the ratio of the equilateral triangle's perimeter to the square's perimeter?

   Answer: $\boxed{1}$

   Solution: The perimeter of the equilateral triangle is $12\cdot 3=36$ and the perimeter of the square is $9\cdot 4=36$, which gives a ratio of $\boxed{1}$.

3. In my drawer, I have 17 unique pairs of socks. If I randomly take socks out of my drawer, how many must I take to guarantee I have a matching pair?

   Answer: $\boxed{18}$

   Solution: Worst case scenario, I have one sock of every pair after drawing 17 socks. My 18th sock must be in a pair with one of the previous 17, so our answer is $\boxed{18}$.

4. Scoorge McDuck accidentally dropped a money bag! Ma Beagle starts 100 meters east of the bag and Flintheart Glomgold starts 150 meters west of the bag. If Glomgold runs at 20 meters per second and reaches the bag 5 seconds later than Ma Beagle, how fast was Ma Beagle going in meters per second?

   Answer: $\boxed{40}$

   Solution: Let's call Ma Beagle's speed in meters per second m. It will take Ma Beagle $\frac{100}{m}$ seconds to reach the bag and Glomgold $\frac{150}{20}=7.5$ seconds to reach the bag. If Glomgold reached the bag 5 seconds later than Ma Beagle, it took Ma Beagle $7.5-5=2.5$ seconds to reach the bag, so $\frac{100}{m}=2.5$ and thus $m=\boxed{40}$ meters per second.

5. Charles is a very indecisive person. He has to choose one out of three books to read. There are 11 copies of the first book, and 15 copies of another. If there are a total of 50 books and Charles is equally likely to choose any of the 50 books to read, the probability that Charles chooses the third type of book is $\frac{a}{b}$ where a and b are relatively prime. What is $a+b$?

   Answer: $\boxed{37}$

   Solution: We know there are $50-11-15=24$ copies of the third book. The probability he will choose a copy of this book is $\frac{24}{50}=\frac{12}{25}$. Therefore, the answer is $12+25=\boxed{37}$.

6. Violet has been rolling on a slope at 50 miles per hour for 5 minutes. If the slope is 10 miles long, what does her average speed need to be during the rest of her trip so that she reaches the end of the slope in exactly 10 minutes?

   Answer: $\boxed{35}$

   Solution: In 5 minutes, the caravan travels $\frac{50}{12}$ miles which means she has $10-\frac{50}{12}$ miles left on the slope. This distance needs to be traveled in 10 minutes. Therefore, her average speed is $\left(10-\frac{50}{12}\right)\cdot 6=\boxed{35}$.

7. The number $\underline{1}\,\underline{2}\,\underline{3}\,\underline{9}\,\underline{1}\,\underline{0}\,\underline{2}\,\underline{4}\,\underline{X}$, where X represents a single digit, is divisible by both 2 and 3. What is the sum of all possible values of X?

   Answer: $\boxed{10}$

   Solution: Using divisibility rules, we have that X must be even so that our number is divisible by 2, and it must also be in the set $\{2,5,8\}$ so that our number is divisibly by 3. The only numbers that satisfy both of these constraints are 2 and 8, so we achieve an answer of $\boxed{10}$.

8. Call a natural number n "magic" if it is a perfect square, divisible by 6, and a multiple of 5. How many magic numbers are there that are less than 10000?

   Answer: $\boxed{3}$

   Solution: Since magic numbers are divisible by 6 and multiples of 5, they must have at least one 2, 3, and 5 in their prime factorizations. They are also perfect squares, so each primes' powers must be even. This means all magic numbers are multiples of $2^2\times 3^2\times 5^2=900$. Any number that is $n^2\cdot 900$, where n is a positive integer, is magic. We see that $n=1,2,3$ work but $n=4$ gives $16\cdot 900=14400>10000$. So, our answer is $\boxed{3}$.

9. Square $ABCD$ has side length 4 and a point P inside it. What is the area of triangle ABP plus the area of CDP?

   Answer: $\boxed{8}$

   Solution: The area of triangle ABP is equal to $\frac{1}{2}\times AB\times d=2d$, where d is the distance from side AB to point P. Then, the area of CDP is equal to $\frac{1}{2}\times CD\times(4-d)=2(4-d)$. So the sum of the areas is $2(d+4-d)=\boxed{8}$.

10. Peter has a collection of foxes and rabbits. He says three statements:

    - The number of foxes is 20 more than the number of rabbits.
    - The number of foxes is equal to the number of rabbits squared.
    - The number of foxes is equal to six times the number of rabbits.

    However, it is revealed that one of Peter's statements is false, while the other two are true. What is the maximum number of foxes Peter has?

    Answer: $\boxed{36}$

    Solution: Let the number of foxes be f and the number of rabbits be r. If the first statement is false, we get that

    $$f=r^2, f=6r \implies f=36, r=6$$

    If the second statement is false, we get that

    $$f=r+20, f=6r \implies f=24, r=4$$

    If the third statement is false, we get that

    $$f=r+20, f=r^2 \implies f=25, r=5$$

    We see that the largest f of these three is $f=\boxed{36}$, when the first statement is false.

11. A number is called "alternative" if its digits alternate between two distinct values such that the preceding and succeeding digit of a number are necessarily the same. As an example, 3737 is an alternative number, but 2494 is not. How many six digit alternative numbers are divisible by 4 if numbers can begin with 0?

    Answer: $\boxed{22}$

    Solution: Using the divisibility condition for 4, it follows that the last two digits have to form a number divisible by 4. This then determines the rest of the number due to the alternating condition. Noting that $00, 44, 88$ do not work, but the rest of the multiples from 0 to 96 are valid, gives an answer of $25 - 3 = \boxed{22}$.

12. Anna, Bob, and Carol play a game where they each select a unique number from $\{1, 2, 3, \ldots, 10\}$. They select a, b, and c, respectively. They then ask questions about each other's numbers, to which their responses are truthful.

    - Anna asks Bob, "Is your number prime?" to which Bob replies No.
    - Bob asks Carol, "Is your number greater than 5?" to which Carol replies Yes.
    - Carol asks Anna, "Is your number a factor of 12?" to which Anna replies Yes.

    How many possible triples (a, b, c) are there?

    Answer: $\boxed{111}$

    Solution: We know that:

    - $a \in \{1, 2, 3, 4, 6\}$
    - $b \in \{1, 4, 6, 8, 9, 10\}$
    - $c \in \{6, 7, 8, 9, 10\}$

    Now, consider the value of a. If a is 1 or 4, then b has 5 options. If b picks 6, 8, 9, or 10, then c will have 4 options. Otherwise, c has 5, so we get a total of $2(4 \cdot 4 + 5) = 42$. If a is 2 or 3, then b has 6 options. From the same argument as before, if b picks 6, 8, 9, or 10, c has 4 options. If not, c has 5, which gives us another $2(4 \cdot 4 + 2 \cdot 5) = 52$. Finally, if $a = 6$, b has 5 options. Like before, if $b = 8, 9, 10$, c has 4 options, otherwise, it has 5. So, we get another $3 \cdot 4 + 5 = 17$. Adding all the cases up gives $42 + 52 + 17 = \boxed{111}$

13. For how many positive integers n less than 1000 is $\text{lcm}(90, n) = 5n$?

    Answer: $\boxed{44}$

    Solution: $90 = 2^1 \times 3^2 \times 5^1$. Consider the prime factorization of n: $n = 2^{e_2} \times 3^{e_3} \times 5^{e_5} \times \ldots$. If we want $(90, n)$ to equal $5n$, we need $\max(1, e_2) = e_2, \max(2, e_3) = e_3$, and $\max(1, e_5) = e_5 + 1$. This means that $e_2 \geq 1, e_3 \geq 2$, and $e_5 = 0$. So n can be any multiple of $2^1 \times 3^2 = 18$ that is not divisible by 5. This yields an answer of $\lfloor \frac{1000}{18} \rfloor - \lfloor \frac{1000}{90} \rfloor = \boxed{44}$.

14. Helen wanted to go to Emmy's birthday party 7 blocks east and 6 blocks north. She also wants to buy some instruments as a gift, but can only find one xylophone dealer 2 blocks east and 3 blocks north. She then decides to buy the rest of the instruments at an instrument shop 5 blocks east and 4 blocks north of her starting point, and finally goes to the party. How many ways can Helen get to Emmy's birthday party passing through the two points if she can only go 1 block north or east at a time?

    Answer: $\boxed{240}$

    Solution: To get to the first point, Helen must go 2 blocks east and 3 blocks north, in any order. This gives $\binom{5}{2}$ ways to get to the first point, as you can order the five total directions in 5! ways and then divide by the number of ways to order the indistinguishable directions (2! and 3!). From there, she has to go 3 blocks east and 1 block north to get to the instrument shop. By the same argument, this is $\binom{4}{1}$. To get to her final destination, she then needs to go 2 blocks east and 2 blocks north, which is $\binom{4}{2}$ ways. Since each three paths are independent, our answer is

$$\binom{5}{2} \cdot \binom{4}{1} \cdot \binom{4}{2} = 10 \cdot 4 \cdot 6 = \boxed{240}$$

15. The integer n is the smallest positive multiple of 24 such that each digit is either 2 or 3. Compute $\frac{n}{24}$.

    Answer: $\boxed{93}$

    Solution: As n is a multiple of 3, the sum of its digits are as well. Since n is also a multiple of 8, the last three digits sum to a multiple of 8. This narrows the last three digits to be 232. To make n a multiple of 3, we can prepend this to a 2, giving us 2232, or an answer of $\boxed{93}$

16. How many 5 letter sequences of the letters A, B, C, and D have an even number of A's? Note that zero is an even number.

    Answer: $\boxed{528}$

    Solution: If there are $0A's$, then we have $3^5 = 243$ working strings. If there are $2A's$, then we have $\binom{5}{2} = 10$ ways to place the $A's$, with an additional $3^3 = 27$ ways to pick the other three digits. If we have $4A's$, then we have $\binom{5}{4} = 5$ ways to place the $A's$ and 3 ways to pick the last digit. This sums to

$$243 + 27 \cdot 10 + 5 \cdot 3 = \boxed{528}$$