Joe Holbrook Memorial Invitational Competition

5th Grade Solutions

March 19, 2023

1. Define $a \oplus b$ to be $a + \dfrac{b}{a+1}$. Compute $(2 \oplus 9) \times (8 \oplus 63)$.

   Answer: $\boxed{75}$

   Solution: We use the definition to get

   $$(2 \oplus 9) \times (8 \oplus 63) = \left(2 + \frac{9}{3}\right) \times \left(8 + \frac{63}{9}\right) = 5 \cdot 15 = \boxed{75}$$

2. There is an equilateral triangle with side length 12 and a square with side length 9. What is the ratio of the equilateral triangle's perimeter to the square's perimeter?

   Answer: $\boxed{1}$

   Solution: The perimeter of the equilateral triangle is $12 \cdot 3 = 36$ and the perimeter of the square is $9 \cdot 4 = 36$, which gives a ratio of $\boxed{1}$.

3. In my drawer, I have 17 unique pairs of socks. If I randomly take socks out of my drawer, how many must I take to guarantee I have a matching pair?

   Answer: $\boxed{18}$

   Solution: Worst case scenario, I have one sock of every pair after drawing 17 socks. My 18th sock must be in a pair with one of the previous 17, so our answer is $\boxed{18}$.

4. Charles is a very indecisive person. He has to choose one out of three books to read. There are 11 copies of the first book, and 15 copies of another. If there are a total of 50 books and Charles is equally likely to choose any of the 50 books to read, the probability that Charles chooses the third type of book is $\dfrac{a}{b}$ where a and b are relatively prime. What is $a + b$?

   Answer: $\boxed{37}$

   Solution: We know there are $50 - 11 - 15 = 24$ copies of the third book. The probability he will choose a copy of this book is $\dfrac{24}{50} = \dfrac{12}{25}$. Therefore, the answer is $12 + 25 = \boxed{37}$.

5. The number $\underline{1}\,\underline{2}\,\underline{3}\,\underline{9}\,\underline{1}\,\underline{0}\,\underline{2}\,\underline{4}\,\underline{X}$, where X represents a single digit, is divisible by both 2 and 3. What is the sum of all possible values of X?

   Answer: $\boxed{10}$

   Solution: Using divisibility rules, we have that X must be even so that our number is divisible by 2, and it must also be in the set $\{2, 5, 8\}$ so that our number is divisibly by 3. The only numbers that satisfy both of these constraints are 2 and 8, so we achieve an answer of $\boxed{10}$.

6. Square $ABCD$ has side length 4 and a point P inside it. What is the area of triangle ABP plus the area of CDP?

Answer: $\boxed{8}$

Solution: The area of triangle ABP is equal to $\frac{1}{2} \times AB \times d = 2d$, where d is the distance from side AB to point P. Then, the area of CDP is equal to $\frac{1}{2} \times CD \times (4-d) = 2(4-d)$. So the sum of the areas is $2(d+4-d) = \boxed{8}$.

7. Violet has been rolling on a slope at 50 miles per hour for 5 minutes. If the slope is 10 miles long, what does her average speed need to be during the rest of her trip so that she reaches the end of the slope in exactly 10 minutes?

Answer: $\boxed{35}$

Solution: In 5 minutes, the caravan travels $\frac{50}{12}$ miles which means she has $10-\frac{50}{12}$ miles left on the slope. This distance needs to be traveled in 10 minutes. Therefore, her average speed is $\left(10-\frac{50}{12}\right) \cdot 6 = \boxed{35}$.

8. Call a natural number n "magic" if it is a perfect square, divisible by 6, and a multiple of 5. How many magic numbers are there that are less than 10000?

Answer: $\boxed{3}$

Solution: Since magic numbers are divisible by 6 and multiples of 5, they must have at least one 2, 3, and 5 in their prime factorizations. They are also perfect squares, so each primes' powers must be even. This means all magic numbers are multiples of $2^2 \times 3^2 \times 5^2 = 900$. Any number that is $n^2 \cdot 900$, where n is a positive integer, is magic. We see that $n = 1, 2, 3$ work but $n = 4$ gives $16 \cdot 900 = 14400 > 10000$. So, our answer is $\boxed{3}$.

9. Peter has a collection of foxes and rabbits. He says three statements:

- The number of foxes is 20 more than the number of rabbits.
- The number of foxes is equal to the number of rabbits squared.
- The number of foxes is equal to six times the number of rabbits.

However, it is revealed that one of Peter's statements is false, while the other two are true. What is the maximum number of foxes Peter has?

Answer: $\boxed{36}$

Solution: Let the number of foxes be f and the number of rabbits be r. If the first statement is false, we get that

$$f = r^2, f = 6r \implies f = 36, r = 6$$

If the second statement is false, we get that

$$f = r + 20, f = 6r \implies f = 24, r = 4$$

If the third statement is false, we get that

$$f = r + 20, f = r^2 \implies f = 25, r = 5$$

We see that the largest f of these three is $f = \boxed{36}$, when the first statement is false.

10. A number is called "alternative" if its digits alternate between two distinct values such that the preceding and succeeding digit of a number are necessarily the same. As an example, 3737 is an alternative number, but 2494 is not. How many six digit alternative numbers are divisible by 4 if numbers can begin with 0?

Answer: $\boxed{22}$

Solution: Using the divisibility condition for 4, it follows that the last two digits have to form a number divisible by 4. This then determines the rest of the number due to the alternating condition. Noting that $00, 44, 88$ do not work, but the rest of the multiples from 0 to 96 are valid, gives an answer of $25-3 = \boxed{22}$.

11. For how many positive integers n less than 1000 is $\text{lcm}(90, n) = 5n$?

Answer: $\boxed{44}$

Solution: $90 = 2^1 \times 3^2 \times 5^1$. Consider the prime factorization of n: $n = 2^{e_2} \times 3^{e_3} \times 5^{e_5} \times \ldots$. If we want $(90, n)$ to equal $5n$, we need $\max(1, e_2) = e_2, \max(2, e_3) = e_3$, and $\max(1, e_5) = e_5 + 1$. This means that $e_2 \geq 1, e_3 \geq 2$, and $e_5 = 0$. So n can be any multiple of $2^1 \times 3^2 = 18$ that is not divisible by 5. This yields an answer of $\lfloor \frac{1000}{18} \rfloor - \lfloor \frac{1000}{90} \rfloor = \boxed{44}$.

12. Helen wanted to go to Emmy's birthday party 7 blocks east and 6 blocks north. She also wants to buy some instruments as a gift, but can only find one xylophone dealer 2 blocks east and 3 blocks north. She then decides to buy the rest of the instruments at an instrument shop 5 blocks east and 4 blocks north of her starting point, and finally goes to the party. How many ways can Helen get to Emmy's birthday party passing through the two points if she can only go 1 block north or east at a time?

Answer: $\boxed{240}$

Solution: To get to the first point, Helen must go 2 blocks east and 3 blocks north, in any order. This gives $\binom{5}{2}$ ways to get to the first point, as you can order the five total directions in 5! ways and then divide by the number of ways to order the indistinguishable directions (2! and 3!). From there, she has to go 3 blocks east and 1 block north to get to the instrument shop. By the same argument, this is $\binom{4}{1}$. To get to her final destination, she then needs to go 2 blocks east and 2 blocks north, which is $\binom{4}{2}$ ways. Since each three paths are independent, our answer is

$$\binom{5}{2} \cdot \binom{4}{1} \cdot \binom{4}{2} = 10 \cdot 4 \cdot 6 = \boxed{240}$$

13. Four circles of radius 4 are internally tangent to a circle of radius r, with each of the smaller circles externally tangent to two others. If r is written as $a + b\sqrt{c}$ for positive integers a, b, c with c squarefree, then find $a + b + c$.

Answer: $\boxed{10}$

Solution: Suppose the circles are c_1, c_2, c_3, and c_4, their respective centers are O_1, O_2, O_3, and O_4, and their intersections with the outer circle are at I_1, I_2, I_3, I_4. WLOG, the circles are arranged so that c_2 is tangent to c_1 and c_3 and across from c_4, and so on. Because the four inner circles have equal radii, by symmetry $\overline{I_1I_3}$ and $\overline{I_2I_4}$ are on the diameters of the larger circle. Hence, $I_1I_3 = I_2I_4 = 2r$. We can write these diameters in terms of the distance between opposite centers by noting that $I_1I_3 = 4 + O_1O_3 + 4$ $= 8 + O_1O_3$. Thus, $2r = 8 + O_1O_3$, so $r = 4 + O_1O_3/2$. (The same goes for I_2I_4.)

We can compute O_1O_3 and O_2O_4 by considering square $O_1O_2O_3O_4$, which has side lengths of $2(4) = 8$ and diagonal lengths $8\sqrt{2}$. Because O_1O_3 and O_2O_4 are diagonals of this square, they must have length $8\sqrt{2}$, so $r = 4 + O_1O_3/2 = 4 + 4\sqrt{2}$ and $4 + 4 + 2 = \boxed{10.}$

14. The integer n is the smallest positive multiple of 24 such that each digit is either 2 or 3. Compute $\frac{n}{24}$.

Answer: $\boxed{93}$

Solution: As n is a multiple of 3, the sum of its digits are as well. Since n is also a multiple of 8, the last three digits sum to a multiple of 8. This narrows the last three digits to be 232. To make n a multiple of 3, we can prepend this to a 2, giving us 2232, or an answer of $\boxed{93}$

15. Let $y > 1$. Given $y + \frac{1}{y} = 34$, $y - \frac{1}{y}$ can be expressed as $a\sqrt{b}$, where b is not divisible by the prime factor of any square. Find $a + b$.

Answer: $\boxed{27}$

Solution: Notice that squaring $y + \frac{1}{y}$ yields $y^2 + \frac{1}{y^2} + 2 = 1156$. The square of $y - \frac{1}{y}$, however, is $y^2 + \frac{1}{y^2} - 2$. So, $(y + \frac{1}{y})^2 - 4 = (y - \frac{1}{y})^2$. Therefore, $(y - \frac{1}{y})^2 = 1152 = 2^7 \times 3^2$. At this point, it is important to use the fact that $y > 1$, as this guarantees $y - \frac{1}{y}$ to be positive. So, $y - \frac{1}{y} = 24\sqrt{3}$. $24 + 3 = \boxed{27}$.

16. Consider the set of all parallelograms that have the points $(1, 2), (2, 5), (-5, 2)$ among its vertices. Compute the sum of the areas of all of these parallelograms.

    Answer: $\boxed{54}$

    Solution: It is well known that there are 3 such parallelograms that satisfy the given conditions. (We can also see this by picking one of the segments to be the diagonal). Furthermore, each of these parallelograms will have double the area formed by the triangle with the given vertices (since the triangle is just one half of the parallelogram), and so our answer is just $3 \cdot 2 = 6$ times the area of the triangle, which can be easily computed using the Shoelace formula to be 9. Thus, the answer is $6 \cdot 9 = \boxed{54}$. We can also note that it has a height of 3 and a base of 6 (formed by the points (1, 2) and (-5, 2)), which also gives $3 \cdot 6 \cdot \frac{1}{2} = 9$.