Joe Holbrook Memorial Invitational Competition

7th Grade Solutions

March 19, 2023

1. Charles is a very indecisive person. He has to choose one out of three books to read. There are 11 copies of the first book, and 15 copies of another. If there are a total of 50 books and Charles is equally likely to choose any of the 50 books to read, the probability that Charles chooses the third type of book is $\frac{a}{b}$ where a and b are relatively prime. What is $a+b$?

   Answer: $\boxed{37}$

   Solution: We know there are $50 - 11 - 15 = 24$ copies of the third book. The probability he will choose a copy of this book is $\frac{24}{50} = \frac{12}{25}$. Therefore, the answer is $12 + 25 = \boxed{37}$.

2. In my drawer, I have 17 unique triplets of socks. If I randomly take socks out of my drawer, how many must I take to guarantee I have a matching triplet?

   Answer: $\boxed{35}$

   Solution: Worst case scenario, I have two socks of every triplet after drawing 34 socks. My 35th sock must be in a triple with one of the previous 34, so our answer is $\boxed{35}$.

3. Square $ABCD$ has side length 4 and a point P inside it. What is the area of triangle ABP plus the area of CDP?

   Answer: $\boxed{8}$

   Solution: The area of triangle ABP is equal to $\frac{1}{2} \times AB \times d = 2d$, where d is the distance from side AB to point P. Then, the area of CDP is equal to $\frac{1}{2} \times CD \times (4 - d) = 2(4 - d)$. So the sum of the areas is $2(d + 4 - d) = \boxed{8}$.

4. Violet has been rolling on a slope at 50 miles per hour for 5 minutes. If the slope is 10 miles long, what does her average speed need to be during the rest of her trip so that she reaches the end of the slope in exactly 10 minutes?

   Answer: $\boxed{35}$

   Solution: In 5 minutes, the caravan travels $\frac{50}{12}$ miles which means she has $10 - \frac{50}{12}$ miles left on the slope. This distance needs to be traveled in 10 minutes. Therefore, her average speed is $\left(10 - \frac{50}{12}\right) \cdot 6 = \boxed{35}$.

5. A number is called "alternative" if its digits alternate between two distinct values such that the preceding and succeeding digit of a number are necessarily the same. As an example, 3737 is an alternative number, but 2494 and 2222 is not. How many six digit alternative numbers are divisible by 4 if numbers can begin with 0?

   Answer: $\boxed{22}$

   Solution: Using the divisibility condition for 4, it follows that the last two digits have to form a number divisible by 4. This then determines the rest of the number due to the alternating condition. Noting that $00, 44, 88$ do not work, but the rest of the multiples from 0 to 96 are valid, gives an answer of $25 - 3 = \boxed{22}$.

6. How many 5 letter sequences of the letters A, B, C, and D have an even number of A's? Note that zero is an even number.

   Answer: $\boxed{528}$

   Solution: If there are $0A's$, then we have $3^5 = 243$ working strings. If there are $2A's$, then we have $\binom{5}{2} = 10$ ways to place the $A's$, with an additional $3^3 = 27$ ways to pick the other three digits. If we have $4A's$, then we have $\binom{5}{4} = 5$ ways to place the $A's$ and 3 ways to pick the last digit. This sums to

$$243 + 27 \cdot 10 + 5 \cdot 3 = \boxed{528}$$

7. Call a natural number n "magic" if it is a perfect square, divisible by 6, and a multiple of 5. How many magic numbers are there that are less than 10000?

   Answer: $\boxed{3}$

   Solution: Since magic numbers are divisible by 6 and multiples of 5, they must have at least one 2, 3, and 5 in their prime factorizations. They are also perfect squares, so each primes' powers must be even. This means all magic numbers are multiples of $2^2 \times 3^2 \times 5^2 = 900$. Any number that is $n^2 \cdot 900$, where n is a positive integer, is magic. We see that $n = 1, 2, 3$ work but $n = 4$ gives $16 \cdot 900 = 14400 > 10000$. So, our answer is $\boxed{3}$.

8. Peter has a collection of foxes and rabbits. He says three statements:

   - The number of foxes is 20 more than the number of rabbits.
   - The number of foxes is equal to the number of rabbits squared.
   - The number of foxes is equal to six times the number of rabbits.

   However, it is revealed that one of Peter's statements is false, while the other two are true. What is the maximum number of foxes Peter has?

   Answer: $\boxed{36}$

   Solution: Let the number of foxes be f and the number of rabbits be r. If the first statement is false, we get that

$$f = r^2, f = 6r \implies f = 36, r = 6$$

   If the second statement is false, we get that

$$f = r + 20, f = 6r \implies f = 24, r = 4$$

   If the third statement is false, we get that

$$f = r + 20, f = r^2 \implies f = 25, r = 5$$

   We see that the largest f of these three is $f = \boxed{36}$, when the first statement is false.

9. For how many positive integers n less than 1000 is $\text{lcm}(90, n) = 5n$?

   Answer: $\boxed{44}$

   Solution: $90 = 2^1 \times 3^2 \times 5^1$. Consider the prime factorization of n: $n = 2^{e_2} \times 3^{e_3} \times 5^{e_5} \times \ldots$. If we want $(90, n)$ to equal $5n$, we need $\max(1, e_2) = e_2, \max(2, e_3) = e_3$, and $\max(1, e_5) = e_5 + 1$. This means that $e_2 \geq 1, e_3 \geq 2$, and $e_5 = 0$. So n can be any multiple of $2^1 \times 3^2 = 18$ that is not divisible by 5. This yields an answer of $\lfloor \frac{1000}{18} \rfloor - \lfloor \frac{1000}{90} \rfloor = \boxed{44}$.

10. Four circles of radius 4 are internally tangent to a circle of radius r, with each of the smaller circles externally tangent to two others. If r is written as $a + b\sqrt{c}$ for positive integers a, b, c with c squarefree, then find $a + b + c$.

    Answer: $\boxed{10}$

Solution: Suppose the circles are c_1, c_2, c_3, and c_4, their respective centers are O_1, O_2, O_3, and O_4, and their intersections with the outer circle are at I_1, I_2, I_3, I_4. WLOG, the circles are arranged so that c_2 is tangent to c_1 and c_3 and across from c_4, and so on. Because the four inner circles have equal radii, by symmetry $\overline{I_1I_3}$ and $\overline{I_2I_4}$ are on the diameters of the larger circle. Hence, $I_1I_3 = I_2I_4 = 2r$. We can write these diameters in terms of the distance between opposite centers by noting that $I_1I_3 = 4 + O_1O_3 + 4 = 8 + O_1O_3$. Thus, $2r = 8 + O_1O_3$, so $r = 4 + O_1O_3/2$. (The same goes for I_2I_4.)

We can compute O_1O_3 and O_2O_4 by considering square $O_1O_2O_3O_4$, which has side lengths of $2(4) = 8$ and diagonal lengths $8\sqrt{2}$. Because O_1O_3 and O_2O_4 are diagonals of this square, they must have length $8\sqrt{2}$, so $r = 4 + O_1O_3/2 = 4 + 4\sqrt{2}$ and $4 + 4 + 2 = \boxed{10.}$

11. Consider the set of all parallelograms that have the points $(1, 2), (2, 5), (-5, 2)$ among its vertices. Compute the sum of the areas of all of these parallelograms.

Answer: $\boxed{54}$

Solution: It is well known that there are 3 such parallelograms that satisfy the given conditions. (We can also see this by picking one of the segments to be the diagonal). Furthermore, each of these parallelograms will have double the area formed by the triangle with the given vertices (since the triangle is just one half of the parallelogram), and so our answer is just $3 \cdot 2 = 6$ times the area of the triangle, which can be easily computed using the Shoelace formula to be 9. Thus, the answer is $6 \cdot 9 = \boxed{54}$. We can also note that it has a height of 3 and a base of 6 (formed by the points (1, 2) and (-5, 2)), which also gives $3 \cdot 6 \cdot \frac{1}{2} = 9$.

12. Nikhil can paint a room in x hours, Jaiden can paint the same room in y hours, and working together, they can paint the same room in xy hours (where x and y are real numbers). What is the maximum amount of minutes it could take them to paint the room together?

Answer: $\boxed{15}$

Solution: We see that $\frac{1}{x} + \frac{1}{y} = \frac{1}{xy} \implies x + y = 1$. We want to minimize $xy = x(1-x) = x - x^2$. It is well known that that a polynomial of the form $ax^2 + bx + c$ is minimized at $x = -\frac{b}{2a} = \frac{1}{2}$ which gives $\frac{1}{4}$ of an hour or $\boxed{15}$ minutes.

13. The integer n is the smallest positive multiple of 24 such that each digit is either 2 or 3. Compute $\frac{n}{24}$.

Answer: $\boxed{93}$

Solution: As n is a multiple of 3, the sum of its digits are as well. Since n is also a multiple of 8, the last three digits sum to a multiple of 8. This narrows the last three digits to be 232. To make n a multiple of 3, we can prepend this to a 2, giving us 2232, or an answer of $\boxed{93}$

14. How many distinct terms are there in the sequence $\left\lfloor \frac{n^2}{2022} \right\rfloor$ as n goes from 1 to 2022? ($\lfloor x \rfloor$ denotes the largest integer less than or equal to x. For example, $\lfloor 20.345 \rfloor = 20$).

Answer: $\boxed{1517}$

Solution: Consider $\frac{(n+1)^2}{2022} - \frac{(n)^2}{2022} = \frac{2n+1}{2022}$. If this difference exceeds 1, $\lfloor \frac{(n+1)^2}{2022} \rfloor$ and $\lfloor \frac{(n)^2}{2022} \rfloor$ are necessarily different. Let $\frac{2n+1}{2022} = 1 \rightarrow 2n + 1 = 2022 \rightarrow n = 1010.5$. Thus, we have two cases. Case 1: $n \leq 1010$. If $n \leq 1010$, the maximum value of $\lfloor \frac{(n)^2}{2022} \rfloor$ is 504. Since for all $n \leq 1010$, the difference between two consecutive n is less than 1, all values $0, 1, 2, \cdots, 504$ are valid and thus there are 505 distinct values for case 1. Case 2: $n \geq 1011$. If $n \geq 1011$, the difference between any two consecutive n is greater than 1, and thus every n must reach a distinct value. Thus the number of distinct values for case 2 is $2022 - 1011 + 1 = 1012$. So, the total number of distinct values is $505 + 1012 = \boxed{1517}$.

15. There are 7 people in a circle passing one ball around. A person cannot pass the ball to someone adjacent to them, and a person cannot pass the ball to themself. How many ways can the ball be passed around so that the ball starts and ends at the same person and is passed a total of 7 times?

    Answer: $\boxed{2258}$

    Solution: Let the people be named Person 1, Person 2, etc in the clockwise direction. Let the value of a pass to someone x people away in the clockwise direction be x. In other words, the pass from Person 1 to Person 4 is 3. Let a_i be the value of the i-th pass. Notice a pass cannot be adjacent or to themself so $2 \le a_i \le 5$ We essentially want

    $$S = a_1 + a_2 + \cdots + a_7 \equiv 0 \bmod 7$$

    with the above restriction. The bounds of a_i mean $S = 14, 21, 28,$ or 35.

    For the $S = 14$ case, there is only one solution: $a_i = 2$.

    For the $S = 21$ case, this is equivalent to finding the number of solutions to

    $$b_1 + b_2 + \cdots b_7 = 7$$

    with $0 \le b_i \le 3$. There's $\binom{13}{6} = 1716$ solutions by stars and bars with no restrictions on b_i. Then, we can subtract off the cases where a b_i is too big.

    - (0, 0, 0, 0, 0, 0, 7): $\frac{7!}{6!} = 7$
    - (0, 0, 0, 0, 0, 1, 6): $\frac{7!}{5!} = 42$
    - (0, 0, 0, 0, 0, 2, 5): $\frac{7!}{5!} = 42$
    - (0, 0, 0, 0, 1, 1, 5): $\frac{7!}{4!2!} = 105$
    - (0, 0, 0, 0, 0, 3, 4): $\frac{7!}{5!} = 42$
    - (0, 0, 0, 0, 1, 2, 4): $\frac{7!}{4!} = 210$
    - (0, 0, 0, 1, 1, 1, 4): $\frac{7!}{3!3!} = 140$

    There is a total of

    $$1716 - (7 + 42 + 42 + 105 + 42 + 210 + 140) = 1128$$

    valid $(a_1, a_2, \cdots, a_7)$ tuples.

    The $S = 28$ case is symmetric to $S = 21$ and $S = 35$ is symmetric to $S = 14$, so our answer is $1128 \cdot 2 + 2 = \boxed{2258}$

16. In triangle ABC, the angle bisector of $\angle BCA$ intersects AB at D and the circumcircle at E. Given that $AB = 33$, $AC = 28$, $BC = 16$, and $CD = 14$, what is BE?

    Answer: $\boxed{24}$

    Solution: By the Angle Bisector Theorem, we know that $AD = \frac{AC}{AC + BC} \cdot AB = 21$. Since $CD = CE$ is an angle bisector, $\angle ACD = \angle ECB$, and since $\angle CAB = \angle CAD$ and $\angle CEB$ are both inscribed angles subtending arc BC, they are equal as well. This means $\triangle CAD$ is similar to $\triangle CEB$, so $BE = AD \cdot \frac{BC}{CD} = 21 \cdot 16/14 = \boxed{24}$.