Joe Holbrook Memorial Invitational Competition

8th Grade Solutions

March 19, 2023

1. In my drawer, I have 17 unique triplets of socks. If I randomly take socks out of my drawer, how many must I take to guarantee I have a matching triplet?

   Answer: $\boxed{35}$

   Solution: Worst case scenario, I have two socks of every triplet after drawing 34 socks. My 35th sock must be in a triple with one of the previous 34, so our answer is $\boxed{35}$.

2. The number $\underline{1}\,\underline{2}\,\underline{3}\,\underline{9}\,\underline{1}\,\underline{0}\,\underline{2}\,\underline{4}\,\underline{X}$, where X represents a single digit, is divisible by both 2 and 3. What is the sum of all possible values of X?

   Answer: $\boxed{10}$

   Solution: Using divisibility rules, we have that X must be even so that our number is divisible by 2, and it must also be in the set $\{2, 5, 8\}$ so that our number is divisibly by 3. The only numbers that satisfy both of these constraints are 2 and 8, so we achieve an answer of $\boxed{10}$.

3. Square $ABCD$ has side length 4 and a point P inside it. What is the area of triangle ABP plus the area of CDP?

   Answer: $\boxed{8}$

   Solution: The area of triangle ABP is equal to $\frac{1}{2} \times AB \times d = 2d$, where d is the distance from side AB to point P. Then, the area of CDP is equal to $\frac{1}{2} \times CD \times (4 - d) = 2(4 - d)$. So the sum of the areas is $2(d + 4 - d) = \boxed{8}$.

4. A number is called "alternative" if its digits alternate between two distinct values such that the preceding and succeeding digit of a number are necessarily the same. As an example, 3737 is an alternative number, but 2494 and 2222 is not. How many six digit alternative numbers are divisible by 4 if numbers can begin with 0?

   Answer: $\boxed{22}$

   Solution: Using the divisibility condition for 4, it follows that the last two digits have to form a number divisible by 4. This then determines the rest of the number due to the alternating condition. Noting that $00, 44, 88$ do not work, but the rest of the multiples from 0 to 96 are valid, gives an answer of $25 - 3 = \boxed{22}$.

5. How many 5 letter sequences of the letters A, B, C, and D have an even number of A's? Note that zero is an even number.

   Answer: $\boxed{528}$

   Solution: If there are $0A's$, then we have $3^5 = 243$ working strings. If there are $2A's$, then we have $\binom{5}{2} = 10$ ways to place the $A's$, with an additional $3^3 = 27$ ways to pick the other three digits. If we have $4A's$, then we have $\binom{5}{4} = 5$ ways to place the $A's$ and 3 ways to pick the last digit. This sums to

$$243 + 27 \cdot 10 + 5 \cdot 3 = \boxed{528}$$

6. Call a natural number n "magic" if it is a perfect square, divisible by 6, and a multiple of 5. How many magic numbers are there that are less than 10000?

   Answer: $\boxed{3}$

   Solution: Since magic numbers are divisible by 6 and multiples of 5, they must have at least one 2, 3, and 5 in their prime factorizations. They are also perfect squares, so each primes' powers must be even. This means all magic numbers are multiples of $2^2 \times 3^2 \times 5^2 = 900$. Any number that is $n^2 \cdot 900$, where n is a positive integer, is magic. We see that $n = 1, 2, 3$ work but $n = 4$ gives $16 \cdot 900 = 14400 > 10000$. So, our answer is $\boxed{3}$.

7. Four circles of radius 4 are internally tangent to a circle of radius r, with each of the smaller circles externally tangent to two others. If r is written as $a + b\sqrt{c}$ for positive integers a, b, c with c squarefree, then find $a + b + c$.

   Answer: $\boxed{10}$

   Solution: Suppose the circles are c_1, c_2, c_3, and c_4, their respective centers are O_1, O_2, O_3, and O_4, and their intersections with the outer circle are at I_1, I_2, I_3, I_4. WLOG, the circles are arranged so that c_2 is tangent to c_1 and c_3 and across from c_4, and so on. Because the four inner circles have equal radii, by symmetry $\overline{I_1I_3}$ and $\overline{I_2I_4}$ are on the diameters of the larger circle. Hence, $I_1I_3 = I_2I_4 = 2r$. We can write these diameters in terms of the distance between opposite centers by noting that $I_1I_3 = 4 + O_1O_3 + 4 = 8 + O_1O_3$. Thus, $2r = 8 + O_1O_3$, so $r = 4 + O_1O_3/2$. (The same goes for I_2I_4.)

   We can compute O_1O_3 and O_2O_4 by considering square $O_1O_2O_3O_4$, which has side lengths of $2(4) = 8$ and diagonal lengths $8\sqrt{2}$. Because O_1O_3 and O_2O_4 are diagonals of this square, they must have length $8\sqrt{2}$, so $r = 4 + O_1O_3/2 = 4 + 4\sqrt{2}$ and $4 + 4 + 2 = \boxed{10.}$

8. Anna, Bob, and Carol play a game where they each select a unique number from $\{1, 2, 3, \ldots, 10\}$. They select a, b, and c, respectively. They then ask questions about each other's numbers, to which their responses are truthful.

   - Anna asks Bob, "Is your number prime?" to which Bob replies No.
   - Bob asks Carol, "Is your number greater than 5?" to which Carol replies Yes.
   - Carol asks Anna, "Is your number a factor of 12?" to which Anna replies Yes.

   How many possible triples (a, b, c) are there?

   Answer: $\boxed{111}$

   Solution: We know that:

   - $a \in \{1, 2, 3, 4, 6\}$
   - $b \in \{1, 4, 6, 8, 9, 10\}$
   - $c \in \{6, 7, 8, 9, 10\}$

   Now, consider the value of a. If a is 1 or 4, then b has 5 options. If b picks 6, 8, 9, or 10, then c will have 4 options. Otherwise, c has 5, so we get a total of $2(4 \cdot 4 + 5) = 42$. If a is 2 or 3, then b has 6 options. From the same argument as before, if b picks 6, 8, 9, or 10, c has 4 options. If not, c has 5, which gives us another $2(4 \cdot 4 + 2 \cdot 5) = 52$. Finally, if $a = 6$, b has 5 options. Like before, if $b = 8, 9, 10$, c has 4 options, otherwise, it has 5. So, we get another $3 \cdot 4 + 5 = 17$. Adding all the cases up gives $42 + 52 + 17 = \boxed{111}$.

9. A positive integer is k-smooth if all of its prime divisors are less than or equal to k. What is the least common multiple of the 5-smooth numbers less than 100?

   Answer: $\boxed{129600}$

   Solution: We need our number to be divisible by 64, 81, and 25, and this is enough because these are the highest powers of 2, 3, and 5 less than 100. (The numbers are 5-smooth, so primes greater than 5 do not divide them.) The answer is $64 * 81 * 25 = \boxed{129600}$.

10. Consider the set of all parallelograms that have the points $(1, 2), (2, 5), (-5, 2)$ among its vertices. Compute the sum of the areas of all of these parallelograms.

Answer: $\boxed{54}$

Solution: It is well known that there are 3 such parallelograms that satisfy the given conditions. (We can also see this by picking one of the segments to be the diagonal). Furthermore, each of these parallelograms will have double the area formed by the triangle with the given vertices (since the triangle is just one half of the parallelogram), and so our answer is just $3 \cdot 2 = 6$ times the area of the triangle, which can be easily computed using the Shoelace formula to be 9. Thus, the answer is $6 \cdot 9 = \boxed{54}$. We can also note that it has a height of 3 and a base of 6 (formed by the points (1, 2) and (-5, 2)), which also gives $3 \cdot 6 \cdot \frac{1}{2} = 9$.

11. Nikhil can paint a room in x hours, Jaiden can paint the same room in y hours, and working together, they can paint the same room in xy hours (where x and y are real numbers). What is the maximum amount of minutes it could take them to paint the room together?

Answer: $\boxed{15}$

Solution: We see that $\frac{1}{x} + \frac{1}{y} = \frac{1}{xy} \implies x + y = 1$. We want to minimize $xy = x(1-x) = x - x^2$. It is well known that that a polynomial of the form $ax^2 + bx + c$ is minimized at $x = -\frac{b}{2a} = \frac{1}{2}$ which gives $\frac{1}{4}$ of an hour or $\boxed{15}$ minutes.

12. The integer n is the smallest positive multiple of 24 such that each digit is either 2 or 3. Compute $\frac{n}{24}$.

Answer: $\boxed{93}$

Solution: As n is a multiple of 3, the sum of its digits are as well. Since n is also a multiple of 8, the last three digits sum to a multiple of 8. This narrows the last three digits to be 232. To make n a multiple of 3, we can prepend this to a 2, giving us 2232, or an answer of $\boxed{93}$.

13. How many pairs of positive integers (a, r) exist such that a and r divides 210^{17} and $\text{lcm}(a, r^2) = \frac{a^2}{r}$?

Answer: $\boxed{1296}$

Solution: For some prime p, let the largest power of p that divides a, r be p^x, p^y. Then we get that

$$max(x, 2y) = 2x - y$$

If $x > 2y$ we get $x = y$, a contradiction. If $x < 2y$ we get $3y = 2x$. So, the possible values of (x, y) are $(0, 0), (3, 2), (6, 4), (9, 6), (12, 8), (15, 10)$. We have 4 primes, so our answer is $6^4 = \boxed{1296}$.

14. How many distinct terms are there in the sequence $\left\lfloor \frac{n^2}{2022} \right\rfloor$ as n goes from 1 to 2022? ($\lfloor x \rfloor$ denotes the largest integer less than or equal to x. For example, $\lfloor 20.345 \rfloor = 20$).

Answer: $\boxed{1517}$

Solution: Consider $\frac{(n+1)^2}{2022} - \frac{(n)^2}{2022} = \frac{2n+1}{2022}$. If this difference exceeds 1, $\lfloor \frac{(n+1)^2}{2022} \rfloor$ and $\lfloor \frac{(n)^2}{2022} \rfloor$ are necessarily different. Let $\frac{2n+1}{2022} = 1 \to 2n + 1 = 2022 \to n = 1010.5$. Thus, we have two cases. Case 1: $n \leq 1010$. If $n \leq 1010$, the maximum value of $\lfloor \frac{(n)^2}{2022} \rfloor$ is 504. Since for all $n \leq 1010$, the difference between two consecutive n is less than 1, all values $0, 1, 2, \cdots, 504$ are valid and thus there are 505 distinct values for case 1. Case 2: $n \geq 1011$. If $n \geq 1011$, the difference between any two consecutive n is greater than 1, and thus every n must reach a distinct value. Thus the number of distinct values for case 2 is $2022 - 1011 + 1 = 1012$. So, the total number of distinct values is $505 + 1012 = \boxed{1517}$.

15. There are 7 people in a circle passing one ball around. A person cannot pass the ball to someone adjacent to them, and a person cannot pass the ball to themself. How many ways can the ball be passed around so that the ball starts and ends at the same person and is passed a total of 7 times?

Answer: $\boxed{2258}$

Solution: Let the people be named Person 1, Person 2, etc in the clockwise direction. Let the value of a pass to someone x people away in the clockwise direction be x. In other words, the pass from Person 1 to Person 4 is 3. Let a_i be the value of the i-th pass. Notice a pass cannot be adjacent or to themself so $2 \leq a_i \leq 5$ We essentially want

$$S = a_1 + a_2 + \cdots + a_7 \equiv 0 \bmod 7$$

with the above restriction. The bounds of a_i mean $S = 14, 21, 28,$ or 35.

For the $S = 14$ case, there is only one solution: $a_i = 2$.

For the $S = 21$ case, this is equivalent to finding the number of solutions to

$$b_1 + b_2 + \cdots b_7 = 7$$

with $0 \leq b_i \leq 3$. There's $\binom{13}{6} = 1716$ solutions by stars and bars with no restrictions on b_i. Then, we can subtract off the cases where a b_i is too big.

- (0, 0, 0, 0, 0, 0, 7): $\frac{7!}{6!} = 7$
- (0, 0, 0, 0, 0, 1, 6): $\frac{7!}{5!} = 42$
- (0, 0, 0, 0, 0, 2, 5): $\frac{7!}{5!} = 42$
- (0, 0, 0, 0, 1, 1, 5): $\frac{7!}{4!2!} = 105$
- (0, 0, 0, 0, 0, 3, 4): $\frac{7!}{5!} = 42$
- (0, 0, 0, 0, 1, 2, 4): $\frac{7!}{4!} = 210$
- (0, 0, 0, 1, 1, 1, 4): $\frac{7!}{3!3!} = 140$

There is a total of

$$1716 - (7 + 42 + 42 + 105 + 42 + 210 + 140) = 1128$$

valid $(a_1, a_2, \cdots, a_7)$ tuples.

The $S = 28$ case is symmetric to $S = 21$ and $S = 35$ is symmetric to $S = 14$, so our answer is $1128 \cdot 2 + 2 = \boxed{2258}$

16. Let $ABCD$ be a trapezoid with $AB \parallel CD$ and $\angle ADC, \angle BCD < 90^\circ$. If the angle bisector $\angle ADC$ intersects side BC at its midpoint, $AD = 5$, and $BC = 7$, then the interval of possible areas for this trapezoid can be written as $(x, y]$. Given that x can be expressed as $\frac{a\sqrt{b}}{c}$, where b is squarefree and a and c are relatively prime, compute $a + b + c$.

Answer: $\boxed{62}$

Solution: Let M be the midpoint of BC, and extend DM to meet line AB at point E. Due to the parallel lines, $\triangle BME \sim \triangle CMD$. Furthermore, since $BM = MC$, these two triangles are in fact congruent. Thus, $CD = BE$. However, since $\angle ADE = \angle EDC = \angle AED$, $\triangle ADE$ isosceles and thus $AD = AE$. So,

$$AD = AE = AB + BE = AB + CD.$$

Note that this condition works in reverse; as long as $AD = AB + CD$, then if we take M to be midpoint and extend MD to point E, we will find that MD is the angle bisector of $\angle ADC$. So, we have reduced the condition in the problem to $AD + CD = 5$.

Now, $[ABCD] = \frac{AB+CD}{2} \cdot h = \frac{5h}{2}$. Thus, we wish to minimize h. If we drop heights from points A and B to segment CD, we can see that

$$CD = \sqrt{25-h^2} + \sqrt{49-h^2} + AB.$$

As long as $\sqrt{25-h^2}+\sqrt{49-h^2} < 5$, there will exist a length for AB such that $AB+CD = 5$. Finally, to solve this inequality, we simply have to move $\sqrt{49-h^2}$ to the other side, square both sides, and simplify; we find that $h > \frac{7\sqrt{51}}{10}$. Thus, our lower bound on $[ABCD]$ is $\frac{5}{2} \cdot \frac{7\sqrt{51}}{10} = \frac{7\sqrt{51}}{4}$. Our answer is then $\boxed{62}$. (This is not part of the problem, but for an extra challenge, what is y?)