

# Joe Holbrook Memorial Math Competition

6th Grade

October 22, 2023

## General Rules

- You will have **75 minutes** to solve **40 questions**. Your score is the number of correct answers.
- Only answers recorded on the answer sheet will be graded.
- This is an individual test. Anyone caught communicating with another student will be removed from the exam and their score will be disqualified.
- Scores will be posted on the website. Please do not forget your ID number, as that will be the sole means of identification for the scores.
- You may use the following aids:
  - Pencil or other writing utensil
  - Eraser
  - Blank scrap paper
- You may not use the following aids:
  - The Internet
  - Books or other written sources
  - Other people
  - Calculator or other computing device
  - Compass
  - Protractor
  - Ruler or straightedge

## Other Notes

- All answers are integers. Make sure you do not make any mistakes when writing your answers, as you will not be given credit if you do so.
- You do not need to write units in your answers.
- Ties will be broken by the number of correct responses to questions 31 through 40. Further ties will be broken by the number of correct responses in the last five questions.

1. The Man has 25 dollars and the Woman has 18 dollars. If the Man gives the woman 7 dollars, how many dollars does the Woman have?
2. James does twenty push-ups a day every day for two weeks. How many push-ups does he do in total?
3. What is the product of  $\frac{77}{4}$  and  $\frac{44}{7}$ ?
4. How many inches longer is 7.7 feet than 2.2 feet?
5. What is the fewest number of coins required to make 37 cents from pennies, nickels, dimes, and quarters?
6. If I were to split 47 cookies evenly between me and my 5 friends, then give the remainder to myself, how many cookies would I have?
7. A square of side length 420 is divided into 49 smaller squares each with the same side length. What is the side length of one of the 49 squares?
8. Help! My pet ducks ran away, and I can't find them! They ran 50 meters in some direction, so I ran 27 meters in a random direction hoping to catch them. What is the greatest possible distance (in meters) between me and my ducks right now?
9. Joy owns only parrots and cats. She forgets how many pets she has, so she counts the number of heads and legs of her pets. There are 70 heads and 270 legs. How many more cats does she have than parrots?
10. Elsa only uses her powers on summer days when it is both hot and sunny, and summer days can only be hot, sunny, or both. There were 95 days of summer: 75 were hot and 37 were sunny. How many days did Elsa use her powers on?
11.  $7!$  can be written as  $2^a \cdot 3^b \cdot 5^c \cdot 7^d$ . Find  $a - b + c - d$ .
12. One of the angles of an isosceles triangle is  $110^\circ$ . What is the measure of one of the other angles, in degrees?
13. How many distinct ways are there to rearrange the letters in "winner"?
14. The wheels on the bus go round and round at a rate of 14 full rotations per second. If the wheels have a radius of  $\frac{3}{\pi}$  feet, how far, in feet, does the bus travel in one minute?
15. A number is called *powerful* if its digits only consist of powers of 2. How many four-digit numbers are powerful?
16. Out of a group of birds, exactly  $\frac{1}{3}$  can fly and the rest cannot. If 5 of the birds that cannot fly are penguins, what is the minimum possible number of birds in the group?
17. What is the sum of the digits of  $10^{2023} - 10^{2006}$ ?
18. Steven has a perfect steel cylinder that weighs 720 grams. He cuts it down into a perfect cone that has the same height and base. How many grams of steel did he remove? (The density of the steel is constant.)
19. What is the units digit of  $7^{2023} + 9^{2023}$ ?
20. The sum of fourteen consecutive odd integers is 1400. Find the lowest of the consecutive odd integers.
21. Anthony has a perfectly spherical balloon he wants to fit into his perfectly cubical box. The volume of the box is  $216 \text{ ft}^3$ . If he makes the biggest spherical balloon that could fit in the box, the difference between the volume of the box and the balloon can be expressed as  $a - b\pi$ , where  $a$  and  $b$  are positive integers. Find  $a + b$ .
22. Larry flips a coin every minute. If it is heads, he writes down 2 on the blackboard. If it is tails, he writes down 0. He does this for six minutes and ends up with a string (for example, 022020 is one such possible string). Suppose  $p$  is the probability that this string is a valid multiple of 12 (no leading 0s). What is  $64p$ ?
23. An equiangular hexagon has 3 sides of length 4 and 3 sides of length 2. If the area of the hexagon can be expressed as  $a\sqrt{b}$  where  $a$  and  $b$  are positive integers and  $b$  is not divisible by the square of any prime number, find  $a + b$ .

24. Let  $x$ ,  $y$ ,  $x + y$ ,  $xy$  form an arithmetic sequence in that order for some nonzero real numbers  $x$  and  $y$ . Compute the value of  $y$ .
25. What is the greatest common divisor of  $8^4 - 1$  and  $4^8 - 1$ ?
26. Piplup created a  $4 \times 4 \times 4$  wooden cube by combining sixty-four  $1 \times 1 \times 1$  wooden cubes. Between any two wooden cubes sharing a face, Piplup used 1 drop of glue. How many drops of glue did Piplup use in total?
27. Jenny is an efficient pizza worker. What is the largest number of pieces she can cut out of the pizza with exactly 6 cuts?
28. Let  $D(n, m)$  denote the number of factors of  $n$  that are multiples of  $m$ . Compute the integer  $k$  for which  $D(k, 24) = D(216, k) > 0$ .
29. Consider the square of lattice points with vertices of  $(0, 0)$ ,  $(0, 4)$ ,  $(4, 4)$  and  $(4, 0)$ . How many isosceles triangles with all vertices on lattice points can be formed with their base on the edge from  $(0, 0)$  to  $(4, 0)$ ?
30. Alex is playing a card game with a standard 52-card deck in which he places cards one by one (whose values are unknown to him) on top of each other in a pile. The game ends and Alex wins as soon as any of the following three conditions are met:
- The card he places has the same value as the first card.
  - He places a queen onto a king.
  - He places a king onto a queen.

Given that Alex placed a queen on his first move, the probability that Alex wins on his third move is  $\frac{m}{n}$ , where  $m$  and  $n$  are relatively prime positive integers. What is  $m + \frac{n}{17}$ ?

31. The graph of  $y = |ax^2 - b|$  intersects the line  $y = 4$  at exactly 3 points. If the the longest distance between two of these three intersections is 4, what is  $a$ ?
32. A *perplexing* number is a square number with the same amount of digits as the number in the units digit. For example, 1 would be a perplexing number as it is a square number, has 1 digit, and the number in the units digit is 1. Find the sum of the first 3 perplexing numbers.
33. Let  $\triangle ABC$  satisfy  $AB = 20$ ,  $BC = 22$  and  $\angle ABC$  is obtuse. Point  $D$  is on the line parallel to  $AB$  passing through  $C$  such that  $BD$  bisects  $\angle ABC$ . Finally, let  $E$  be the point on  $AD$  such that  $CE$  bisects  $\angle BCD$ . Given that  $AE = 1$ , compute the length of  $DE$ .
34. Suppose  $r$ ,  $s$ , and  $t$  are roots to the equation  $x^3 + 22x^2 - 14x + 2 = 0$ . Find  $\left(\frac{1}{r} + \frac{1}{s} + \frac{1}{t}\right) \left(\frac{1}{rs} + \frac{1}{rt} + \frac{1}{st}\right)$ .
35. I start two stopwatches A and B at the same time. Stopwatch A displays the amount of time elapsed in minutes and seconds like a digital clock, while stopwatch B displays only the total number of seconds elapsed. For example, after 107 seconds have elapsed, stopwatch A displays 1:47 while stopwatch B displays 107. I leave the the two stopwatches sitting for over a minute. When I come back to them, the time shown is such that the digits displayed on timer A can be rearranged to give the digits displayed on timer B. When this happens, what is the number (in seconds) that is displayed on timer B?
36. Alice and Bob have a secret code that could potentially end the world! Luckily, we managed to uncover their secret messages, which were encrypted by replacing every digit with its own respective letter.
- Bob: What was the code again? I know it is either BEE, EBE, or EEB, but I forgot which one.
  - Alice: If you remembered the code was divisible by E, you could've realized it was BEE without asking!
  - Bob: Sorry, I only remembered that the product of the digits of the code was SET, which didn't help...

What is the secret code?

37. When the number  $555\dots555_6$  with 2023 occurrences of the digit 5 is converted to base 36, the sum of the digits of the resulting base 36 number is  $S$ . Find  $S$ .

38. Given that  $x = 2023^{2023}$ , find the minimum possible value of

$$100 \left( \frac{x}{y} + \frac{x^4 + y^4}{4x^2y^2} + \frac{y}{x} \right).$$

39. A sphere of radius 1 is inscribed in a cube, and a line segment connects one point where a face of the cube is tangent to the sphere to a vertex on the opposite side of the cube. If  $\frac{a}{b}$  of the segment's total length lies outside the sphere, where  $a$  and  $b$  are relatively prime positive integers, compute  $a + b$ .

40. If the maximal value of  $5x^4 - x^6$  is  $\frac{a}{b}$  where  $a$  and  $b$  are relatively prime positive integers, find  $a + b$ .