Joe Holbrook Memorial Math Competition

8th Grade

October 22, 2023

General Rules

- You will have **75 minutes** to solve **40 questions**. Your score is the number of correct answers.
- Only answers recorded on the answer sheet will be graded.
- This is an individual test. Anyone caught communicating with another student will be removed from the exam and their score will be disqualified.
- Scores will be posted on the website. Please do not forget your ID number, as that will be the sole means of identification for the scores.
- You may use the following aids:
  - Pencil or other writing utensil
  - Eraser
  - Blank scrap paper
- You may not use the following aids:
  - The Internet
  - Books or other written sources
  - Other people
  - Calculator or other computing device
  - Compass
  - Protractor
  - Ruler or straightedge

Other Notes

- All answers are integers. Make sure you do not make any mistakes when writing your answers, as you will not be given credit if you do so.
- You do not need to write units in your answers.
- Ties will be broken by the number of correct responses to questions 31 through 40. Further ties will be broken by the number of correct responses in the last five questions.

1. Given that $17 \times 37 = 629$, what is the value of 170×370?
2. What is the value of $(2^2 + 0^2 + 2^2 + 3^2)^2(2 + 0 + 2 + 3)$?
3. Compute $\frac{17x^3}{(2x)^2}$ when $x = 8$.
4. What is the fewest number of coins required to make 37 cents from pennies, nickels, dimes, and quarters?
5. Every day, the height of Mr. Tree increases by the same amount. Yesterday, Mr. Tree was 87 feet tall, and today, Mr. Tree is 100 feet tall. How tall will Mr. Tree be tomorrow, in feet?
6. If 2 dimes weigh the same as 1 penny, how many times heavier is a dollar in pennies than a dollar in dimes?
7. Ryan loves wacky measurements. He is 5 feet tall and finds a field that is 2 furlongs long (A furlong is 220 yards). How many Ryans would it take to cover the length of the field?
8. Help! My pet ducks ran away, and I can't find them! They ran 50 meters in some direction, so I ran 27 meters in a random direction hoping to catch them. What is the greatest possible distance (in meters) between me and my ducks right now?
9. 7! can be written as $2^a \cdot 3^b \cdot 5^c \cdot 7^d$. Find $a - b + c - d$.
10. The Fibonacci Sequence is comprised of numbers such that each number is the sum of the previous 2 numbers: $1, 1, 2, 3, 5, 8, 13, 21\ldots$ Consider the first 15 terms of this sequence. If x is the sum of the even terms and y is the sum of the odd terms, find $x - y$.
11. If $10 \leq a \leq 15$ and $-3 \leq b \leq 7$, what is the sum of the minimum and maximum values of $a - b$?
12. Klaus is hypnotized and can only break out of his trance if the code word is said. The code word is four letters long. It starts with one consonant, which is followed by one vowel, which is followed by a different consonant, and which is followed by a different vowel. If y is considered to be a consonant, how many code words are possible?
13. Jeff flips a fair coin 2023 times. If at least half of the coin flips are heads, Jeff will win a prize. The probability that Jeff wins a prize can be expressed as $\frac{a}{b}$, expressed in simplest form. Compute $a + b$.
14. There are three positive integers written on three pieces of paper, and one of the integers is the sum of the other two. Knowing this, Shining looks at the integers on the first two pieces of paper, and remarks, "The integer on the third piece of paper must be 46!" What is the number on the first piece of paper?
15. Given $a + b = 7$ and $ab = 13$, compute $a^3 + b^3$.
16. Two circles of radius $\frac{12}{\pi}$ are overlapped such that the center of each circle is on the other circle's boundary. What is the perimeter of the region of points covered by at least one of the two circles?
17. The Quagmires are making a pancake tower. This tower is made of 3 pancakes with radii 6 inches, 4 inches, and 2 inches stacked, centered, and in that order. Each pancake is a cylinder with a height of 1 inch. If the Quagmires cover the pancake in syrup (not including the bottom of the pancakes), the Quagmires need $a\pi$ square inches of syrup. What is a?
18. What is the sum of all integers x such that $2xy + 7x - 8y = 0$, for some integer y?
19. An equiangular hexagon has 3 sides of length 4 and 3 sides of length 2. If the area of the hexagon can be expressed as $a\sqrt{b}$, where a and b are positive integers and b is not divisible by the square of any prime number, find $a + b$.
20. In some numbers, the number of times a digit shows up in the number is the value of the digit. For example, in 22333, 2 shows up 2 times and 3 shows up 3 times. How many of these numbers exist if they must be 7 digits long? 5255525 is one of these.
21. Masha the mongoose lives on a path around the edge of a circular park with a radius of 6 kilometers. Starting at home, Masha walks the path for 8π kilometers, then decides to cut through the park to walk back home in a straight line. If Masha had to walk $a\sqrt{b}$, where a and b are positive integers and b is not divisible by the square of any prime number, kilometers to get home, find $a + b$.

22. A square of unknown side length has one vertex located at $(0, 0)$ and another vertex located at $(6, 8)$. Find the positive difference between the maximum and minimum possible area of the square.

23. If $f(x) = -x^2 + 7x + 33$ and $g(x) = -x^2 + x + 9$, what is the maximum value of $f(x) + g(x)$?

24. Define "pushing" a number the act of cycling each digit one place to the right. For example, pushing the number 10 twice will give $10 \implies 01 = 1 \implies 1$ and pushing the number 210 twice will give $210 \implies 021 = 21 \implies 12$. Note that a leading-zero digit will be discarded. How many positive numbers less than $1,000,000$, when pushed three times, will return to itself?

25. How many odd numbers can be expressed as a sum of some number of elements in the set $\{0, 1, 2, 4, 8, 16, 32, 64\}$?

26. The graph of $y = |ax^2 - b|$ intersects the line $y = 4$ at exactly 3 points. If the the longest distance between two of these three intersections is 4, what is a?

27. Two circles ω_1 and ω_2 are externally tangent to each other and to a line l, such that circles ω_1 and ω_2 all lie on one side of line l. The centers of ω_1 and ω_2 are A and B, respectively, and the point at which ω_1 and ω_2 are tangent to l are P and Q, respectively. The line AB intersects l at C. The radii of ω_1 and ω_2 are 4 and 9, respectively. If the area of $\triangle ACP$ is $\frac{a}{b}$, expressed in simplest form, then find $a + b$.

28. I start two stopwatches A and B at the same time. Stopwatch A displays the amount of time elapsed in minutes and seconds like a digital clock, while stopwatch B displays only the total number of seconds elapsed. For example, after 107 seconds have elapsed, stopwatch A displays 1:47 while stopwatch B displays 107. I leave the the two stopwatches sitting for over a minute. When I come back to them, the time shown is such that the digits displayed on timer A can be rearranged to give the digits displayed on timer B. When this happens, what is the number (in seconds) that is displayed on timer B?

29. If a, b, and c are integers from 1 to 9 (inclusive) such that $b - \dfrac{a}{b-c}$ is maximized, find $100a + 10b + c$.

30. A *perplexing* number is a square number with the same number of digits as the number in the units digit. For example, 1 would be a perplexing number as it is a square number, has 1 digit, and the number in the units digit is 1. Find the sum of the first 3 perplexing numbers.

31. When the number $555...555_6$ with 2023 occurrences of the digit 5 is converted to base 36, the sum of the digits of the resulting base 36 number is S. Find S.

32. How many subsets of positive integers from $1 - 17$ have the least common multiple of all the numbers in the subset as its largest element?

33. Suppose r, s, and t are roots to the equation $x^3 + 22x^2 - 14x + 2 = 0$. Find $\left(\frac{1}{r} + \frac{1}{s} + \frac{1}{t}\right)\left(\frac{1}{rs} + \frac{1}{rt} + \frac{1}{st}\right)$.

34. In quadrilateral $ABCD$, $AB = 10$, $BC = 6$, $CD = 14\sqrt{2}$, and $\angle C = \angle D = 45^\circ$. Find the area of $ABCD$.

35. We start with a string of n ones, and insert some positive number of plus signs between them to create an expression. For example, if we start with 3 ones, we can form $11 + 1$, $1 + 1 + 1$, or $1 + 11$, evaluating to 12, 3, and 12, respectively. For how many positive integer values of n can we create an expression that evaluates to 123?

36. Alice and Bob have a secret code that could potentially end the world! Luckily, we managed to uncover their secret messages, which were encrypted by replacing every digit with its own respective letter.

    - Bob: What was the code again? I know it is either BEE, EBE, or EEB, but I forgot which one.
    - Alice: If you remembered the code was divisible by E, you could've realized it was BEE without asking!
    - Bob: Sorry, I only remembered that the product of the digits of the code was SET, which didn't help...

    What is the secret code?

37. Find the number of 7 digit sequences comprising the digits 1, 4, 7 (not necessarily all used), such that the string 17 never appears.

38. A square is inscribed in a circle. Chord $\overline{AB}$ of the circle intersects the square at points X and Y. If $AX = XY = YB = 6$, what is the area of the square?

39. A sphere of radius 1 is inscribed in a cube, and a line segment connects one point where a face of the cube is tangent to the sphere to a vertex on the opposite side of the cube. If $\frac{a}{b}$, expressed in simplest form, of the segment's total length lies outside the sphere, compute $a + b$.

40. I recently found a box of 15 used batteries in my attic. Some, all, or none of them are working, and some, all, or none of them are dead. I also found a flashlight that requires two working batteries to turn on. I tried to deduce the number of working batteries by inserting each of the 105 possible pairs of batteries into the flashlight in a random order, recording only the running total for the number of times the flashlight has turned on. Strangely, after testing 95 of the pairs, I still cannot determine how many of them are working just by looking at my running total. At that point, what is the greatest possible running total I recorded?