Answer Key

Name: GRADE 6 KEY **ID**: GRADE 6 KEY

1 . 25	11 . 2	21 . 252	31 . 2
2 . 280	12 . 35	22 . 6	32 . 2469
3 . 121	13 . 360	23 . 16	33 . 11
4 . 66	14 . 5040	24 . 4	34 . 77
5 . 4	15 . 256	25 . 15	35 . 590
6 . 12	16 . 9	26 . 144	36 . 688
7 . 60	17 . 153	27 . 22	37 . 35390
8 . 77	18 . 480	28 . 72	38 . -150
9 . 60	19 . 2	29 . 36	39 . 4
10 . 17	20 . 87	30 . 47	40 . 527

FOR GRADER USE ONLY:

Score 1	Score 2	Score 3	Score 4

Total Score:

Joe Holbrook Memorial Math Competition

6th Grade

October 22, 2023

1. The Woman gains 7 dollars and originally has 18. Therefore, she now has $18 + 7 = \boxed{25}$ dollars.
2. There are 7 days per week, so 2 weeks consist of 14 days. Each day, James does 20 pushups, so we can multiply 14×20 to get a total of $\boxed{280}$ pushups.
3. Their product equals $\frac{77}{4} \times \frac{44}{7} = \frac{77}{7} \times \frac{44}{4} = 11 \times 11 = \boxed{121}$.
4. The difference between 7.7 feet and 2.2 feet is 5.5 feet, or $5.5 \times 12 = \boxed{66}$ inches.
5. To use the least number of coins, we should use as many large-valued coins as possible. Thus, to make 37 cents, we would use 1 quarter, 1 dime, and 2 pennies, for a total of $\boxed{4}$ coins.
6. First, 47 cookies are split between 6 people. Since $47 \div 6 = 7$ remainder 5, this means each person receives 7 cookies, and 5 cookies remain. These 5 cookies are added to the 7 cookies I already have, giving me a total of $7 + 5 = \boxed{12}$ cookies.
7. The area of the original square is 420×420, so each of the smaller squares must have an area of $\frac{420 \times 420}{49} = \frac{420 \times 420}{7 \times 7} = 60 \times 60$. Thus, each of the smaller squares must have a side length of $\boxed{60}$.
8. The worst possible situation is that I ran in the opposite direction as my ducks, in which case the distance between us would be $50 + 27 = \boxed{77}$ meters.
9. We can represent this with the system of equations, where p is the number of parrots and c is the number of cats.
$$p + c = 70 \implies 3p + 3c = 210$$
$$2p + 4c = 270$$
Subtracting the second equation from the first yields $c - p = \boxed{60}$.
10. 75+37 = 112 days, which is more than the number of days in summer. This means that some days are being overcounted – they were both hot AND sunny, so they got counted in both numbers and thus more days of summer were counted than the actual number of days. So, the number of overcounted days is 112-95 = 17 overcounted days. Thus, there were $\boxed{17}$ days that were both hot and sunny when Elsa could have used her powers.
11. Writing out 7!, we have $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7$. Writing out the prime factorizations, we have $7! = 1 \cdot 2 \cdot 3 \cdot 2^2 \cdot 5 \cdot (2 \cdot 3) \cdot 7 = 2^4 \cdot 3^2 \cdot 5^1 \cdot 7^1$. So $a = 4, b = 2, c = 1$, and $d = 1$, so $a - b + c - d = 4 - 2 + 1 - 1 = \boxed{2}$.
12. All three angles of the triangle are x, x, and 110, which should sum to 180. So $2x + 110 = 180$, implying that $x = \boxed{35}$ degrees.
13. Notice that there would be 6! permutations if the two occurrences of the letter n were distinct. However, since the two occurrences of letter n are the same, we have to divide by 2 to correct for overcounting. Therefore, there are $\frac{6!}{2!} = \boxed{360}$ permutations.
14. The circumference of a wheel is $(2\pi) \times (\frac{3}{\pi}) = 6$ feet, so the bus travels $14 \times 6 = 84$ feet per second. In one minute, the bus therefore travels $84 \times 60 = \boxed{5040}$ feet.
15. Such *powerful* numbers can consist of the digits 1, 2, 4, and 8. Thus, we have four choices for each digit in a four-digit number, meaning that there are 4*4*4*4 powerful four-digit numbers. Thus, there are $\boxed{256}$ powerful four-digit numbers.

16. Since $\frac{1}{3}$ of the birds can fly, the total number of birds must be a multiple of 3. $\frac{2}{3}$ of the number of birds must also be greater than or equal to 5 implying the total number of birds must be greater than or equal to $\frac{15}{2}$. The smallest number that satisfies both conditions is $\boxed{9}$.

17. This problem is easiest when you factor 10^{2006} out (as it will only add zeroes on the end). $10^{17} - 1$ will have 17 9's, meaning the answer is $17 \times 9 = \boxed{153}$.

18. The formula for the volume of a cylinder is $\pi r^2 h$ and the formula for the volume of a cone is $\frac{1}{3}\pi r^2 h$, so the cone has a volume $\frac{1}{3}$ that of the cylinder. Since the steel is of constant density, the new mass must be $\frac{1}{3} \cdot 720 = 240$ grams, and the answer is $720 - 240 = \boxed{480}$.

19. The pattern for the units digit of 7^x follows $7, 9, 3, 1, 7\ldots$ Since these digits occur in a cycle of 4, we must find the remainder when 2023 is divided by 4. This remainder is 3, so the units digit of 7^{2023} is 3. The pattern for 9^x is $9, 1, 9, \ldots$ So, we should check whether the exponent is even or odd to see if it is 9 or 1. Since the exponent is odd, the units digit of 9^{2023} is 9. Thus, we can sum these two numbers for $3+9 = 12$, giving the units digit as $\boxed{2}$.

20. Let x be the first term of the arithmetic series. Because each term after the first is 2 greater than the previous, the last term must be $x + 13 \cdot 2 = x + 26$. So, the sum of the arithmetic series, given by the average of the first and last term multiplied by the number of terms, is $\frac{x + x + 26}{2} \cdot 14 = 14x + 14 \cdot 13$. This sum is also given to be 1400. So, we solve the linear equation $14x + 14 \cdot 13 = 1400$. First, we can divide both sides by 14: $x + 13 = 100$. Then, we subtract 13 from both sides to get $x = \boxed{87}$.

21. If Anthony's box is 216 cubic feet, a side of his box is $\sqrt[3]{216}$ feet, or 6 feet long. The radius of the balloon's volume will be half of this or 3 feet, and the volume of the balloon $= \frac{4}{3}\pi(3)^3 = 36\pi$. Thus, the difference between the box's volume and the balloon's volume will be $216 - 36\pi$, and our answer will be $216 + 36 = \boxed{252}$.

22. Note that the first digit must be a 2 for it to be valid, and the last two digits must be 20 or 00 for it to be a multiple of 4. There are few enough possibilities that we can just write them all: 222000, 220200, 202200, 200220, 202020, and 220020, for a total of 6. Each of these has a $\frac{1}{2^6} = \frac{1}{64}$ chance of happening, so the answer is simply $\boxed{6}$.

23. Notice that because the hexagon is equiangular, each angle must be 120 degrees. Now, we can inscribe this hexagon in a rectangle, with the opposite sides of 2 and 4 lying on opposite sides of the rectangle. To find the area of the hexagon, we can subtract the area of the four corner triangles, which are $30-60-90$ triangles because of the measure of each angle in the hexagon. The triangles with hypotenuse 4 will have area $2\sqrt{3}$ each, and the triangles with hypotenuse 2 have area $\frac{\sqrt{3}}{2}$ each. The total rectangle will have an area of $(2\sqrt{3}+\sqrt{3})(1+4+1) = 18\sqrt{3}$. Subtracting the triangle areas, we obtain $18\sqrt{3}-4\sqrt{3}-\sqrt{3} = 13\sqrt{3}$, giving an answer of $13 + 3 = \boxed{16}$.

24. Notice that since the first two terms of the arithmetic sequence are x and y, we have that the common difference of the sequence must be $y-x$. Therefore, we must have $2y-x = x+y$ and $2y = xy$. Rearranging terms in the first equation gives $y = 2x$, and substituting this into the second equation gives $4x = xy$. Since x is nonzero, we divide both sides by x to get that $y = \boxed{4}$.

25. The former factors into $(2^3 - 1)(2^3 + 1)(2^6 + 1) = 3^2 \cdot 5 \cdot 7 \cdot 13$, and the latter factors into $(2^4 - 1)(2^4 + 1)(2^8 + 1) = 3 \cdot 5 \cdot 17 \cdot 257$. This results in a GCD of $\boxed{15}$.

26. Observe that every face of every $1 \times 1 \times 1$ cube is in one of two positions: touching a drop of glue, or facing the outside of the $4 \times 4 \times 4$ cube. There are $6 \times 4^2 = 96$ faces on the outside of the $4 \times 4 \times 4$ cube and $6 \times 4^3 = 384$ faces in total, so $384 - 96 = 288$ faces are touching a drop of glue. Since every drop of glue can be associated with 2 faces touching a drop of glue, there must be $288 \div 2 = \boxed{144}$ drops of glue.

27. We can work recursively. Let p_n be the maximum number of pieces that can be created with n cuts. Notice that to maximize the number of pieces created after the $n + 1$th cut, the $n + 1$th line must cross all other n lines inside the circle, but not pass through any pairwise intersections between lines.

As the $n+1$th line will pass through each of the n line segments, it gets split into $n+1$ segments. Each of these segments splits a piece that was already formed by n cuts in two, thus adding $n+1$ pieces.

So, the base case must be with $n = 1$, which can only split the pizza into two slices. Thus, $p_2 = p_1 + 2 = 3$. Continuing, we obtain that $p_6 = 2 + 2 + 3 + 4 + 5 + 6 = \boxed{22}$.

28. Notice that for $D(n, m) > 0$, n must a multiple of m, meaning that for $D(k, 24) = D(216, k) > 0$, k must be a multiple of 24 and a factor of 216. The only numbers k that are both multiples of $24 = 2^3 \cdot 3$ and $216 = 2^3 \cdot 3^3$ are $2^3 \cdot 3 = 24$, $2^3 \cdot 3^2 = 72$ and $2^3 \cdot 3^3 = 216$. Testing all 3 options yields $\boxed{72}$ as the answer.

29. We proceed by casework based on the type of isosceles triangle. Suppose one of the equal sides of the triangle has its base on the x-axis. Now, we can do casework depending on what the side length of this triangle is. If it is 1, there are 8 of such triangles, and if it is 2, there are 6, and if it is 3, there are 4 of such triangles, and if the side length is 4, there are 2. Thus, there are $8 + 6 + 4 + 2 = 20$ of such triangles. Our second case is if the base of the isosceles triangle falls on the x-axis. Therefore, the length of the base must be even, in order to have the third vertex be on a lattice point with a x coordinate directly centered. So, we begin having a base of length 2. There are 3 such segments, and each have 4 triangles corresponding to the segment, making 12 triangles total. If the length of the base is 4, there is only one such base, and 4 triangles. So, there are $12 + 4 = 16$ triangles in this case. In total, we have $\boxed{36}$ isosceles triangles.

30. Since Alex did not win on his second move, he had to have placed a card that was neither a king nor a queen. Since one queen was already placed there are 44 of the remaining 51 cards that satisfy this condition. To win on his third move, he must place a card with the same value as on his second, meaning that there are only 3 of the remaining 50 cards would satisfy the condition. So, $\frac{m}{n} = \frac{44}{51} \cdot \frac{3}{50} = \frac{22}{17 \cdot 25}$. So, $m + n/17 = 22 + 25 = \boxed{47}$.

31. The horizontal line intersects the vertex of the parabola. Therefore, $b = 4$. The x-coordinates of the other two intersection points can be solved with $b = ax^2 - b$ or $x = \pm\sqrt{\frac{2b}{a}} = \pm\frac{2\sqrt{2}}{\sqrt{a}}$. The distance between these two points is $\frac{4\sqrt{2}}{\sqrt{a}} = 4$ so $a = \boxed{2}$.

32. All square numbers will have a units digit as one of $\{1, 4, 5, 6, 9\}$. As we already know 1 is perplexing, and no other one-digit squares have a 1 in the units digit, we need to find the first two 4-digit squares which have a 4 in the units digit. 1024 is perplexing, as $32 * 32 = 1024$, and 1444 is perplexing, as $38 * 38 = 1444$. Therefore, our answer is $1 + 1024 + 1444 = \boxed{2469}$.

33. Let the extension of CE intersect the extension of AB at F. By doing some angle chasing, we find that $\triangle BCD$ is isosceles, giving $CD = 22$. In addition, we also get that $\triangle CEF$ is isosceles, giving $BF = 22$. This means that $AF = 2$. Finally, we also get that $\triangle AFE \sim \triangle DCE$, giving that $DE = \frac{CD \cdot AE}{AF} = \frac{22}{2} = \boxed{11}$.

34. By Vieta's, $r + s + t = -22$, $rs + rt + st = -14$, $rst = -2$. $\frac{1}{r} + \frac{1}{s} + \frac{1}{t} = \frac{st + rt + rs}{rst} = \frac{-14}{-2} = 7$. $\frac{1}{rs} + \frac{1}{rt} + \frac{1}{st} = \frac{r + s + t}{rst} = \frac{-22}{-2} = 11$. So the answer is $7 \cdot 11 = \boxed{77}$.

35. Assume that stopwatch A is displaying digits $X{:}YZ$ when we stop it. The number of seconds that elapsed is going to be $60X + 10Y + Z$. Taking this (mod 9), we have the value $6X + Y + Z$.

    However, as the digits can be rearranged to give the digits of B, B is congruent to $X + Y + Z$ (mod 9). Since the quantities $6X + Y + Z$ and $X + Y + Z$ are both 0 (mod 9), their difference will also be 0 (mod 9). So, $(6X + Y + Z) - (X + Y + Z) = 5X$. Therefore, $5X$ must be divisible by 9, so X is also divisible by 9.

    Taking the least possible value of X, 9, we have that the total number of seconds elapsed, the reading on B, is between 540 and 599, and the reading on A begins with a 9. Additionally, the rightmost digit will be the same for both A and B, as $60X + 10Y + Z$ is congruent to Z (mod 10).

    Therefore, the only possible times in this case are of the form $59Z$ seconds, displaying $9 : 5Z$ on stopwatch A, with the smallest possible value being $\boxed{590}$.

36. Notice that E being a divisor of BEE, EBE, or EEB is equivalent to E being a divisor of B00, B0, or B, respectively. Thus, E is a divisor of B00 but not B0, implying that either E equals 4 and B is odd, or E equals 8 and B is either 2 or 6. So the only options are:

$$\text{BEE} = 144,\ 344,\ 544,\ 744,\ 944,\ 288,\ 688.$$

The product of the digits of these numbers, respectively, are:

$$(144, 16),\ (344, 48),\ (544, 80),\ (744, 102),\ (944, 144),\ (288, 128),\ (688, 384).$$

The only product of digits that matches "SET" is the one corresponding to the secret code being $\boxed{688}$.

37. Notice that the number $555...555_6$ with 2023 occurrences of the digit 5 is equal to $6^{2023} - 1$ in base 10. In addition, notice that $6^{2023} = 6 \cdot (6^2)^{1011} = 6000...000_{36}$, where there are 1011 occurrences of the digit 0. If we let D denote the digit for 35 in base 36, we then have that $6^{2023} - 1 = 5DDD...DDD_{36}$, where there are 1011 occurrences of the digit D. We want the digit sum of this base 36 integer, which is $5 + 1011 \cdot 35 = \boxed{35390}$.

38. Notice that

$$100\left(\frac{x}{y} + \frac{x^4+y^4}{4x^2y^2} + \frac{y}{x}\right) = 100\left(\frac{x^4+4x^3y+4y^3x+y^4}{4x^2y^2}\right) = 100\left(\frac{(x+y)^4}{4x^2y^2} - \frac{3}{2}\right).$$

Then, notice that $\frac{(x+y)^4}{4x^2y^2} = \left(\frac{(x+y)^2}{2x^2y^2}\right)^2$. Therefore, because this expression is a square, its minimum is 0 and is achieved when $x = -y$, since the numerator would be 0. Thus, the minimum value is $100\left(\frac{(x+y)^4}{4x^2y^2} - \frac{3}{2}\right) = 100\left(-\frac{3}{2}\right) = \boxed{-150}$.

39. Taking the diagonal cross section of the cube, we get a rectangle with length $2\sqrt{2}$ and width 2. The circle sits in the middle tangent to the top and the bottom. Setting the length outside the circle to x. We use Power of a Point to get $2 = x \cdot \sqrt{6}$. We then get a ratio of $\frac{\frac{2}{\sqrt{6}}}{\sqrt{6}} = \frac{1}{3}$ giving an answer of $\boxed{4}$.

40. Note that $5x^4 - x^6 = x^4(5 - x^2)$. If we let $y = x^2$, the problem is equivalent to finding the maximum of $k = y^2(5 - y)$ where $y \geq 0$. We can use AM-GM for this: $2k = y^2(10 - 2y)$ and $\sqrt[3]{y \cdot (10 - 2y)} \leq \frac{y + y + 10 - 2y}{3} = \frac{10}{3} \implies y^2(10-2y) \leq \frac{1000}{27} \implies y^2(5-y) \leq \frac{500}{27}$. So the answer is $500 + 27 = \boxed{527}$. The equality case is when the three terms y, y, $10 - 2y$ are equal, so when $y = x^2 = \frac{10}{3}$.