Answer Key

Name: GRADE 8 KEY **ID**: GRADE 8 KEY

1 . 62900	11 . 21	21 . 9	31 . 35390
2 . 2023	12 . 8400	22 . 50	32 . 111
3 . 34	13 . 3	23 . 50	33 . 77
4 . 4	14 . 23	24 . 1494	34 . 74
5 . 113	15 . 70	25 . 64	35 . 14
6 . 20	16 . 32	26 . 2	36 . 688
7 . 264	17 . 60	27 . 101	37 . 987
8 . 77	18 . 16	28 . 590	38 . 450
9 . 2	19 . 16	29 . 989	39 . 4
10 . 0	20 . 169	30 . 2469	40 . 45

FOR GRADER USE ONLY:

Score 1	Score 2	Score 3	Score 4

Total Score:

Joe Holbrook Memorial Math Competition

8th Grade

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1. Observe that $170 \times 370 = 17 \times 10 \times 37 \times 10 = 17 \times 37 \times 10 \times 10 = 629 \times 10 \times 10 = \boxed{62900}$, since multiplying a number by 10 is equivalent to appending a 0 to the end of it.
2. We can compute directly to find that the expression equals $17^2 \cdot 7 = \boxed{2023}$.
3. The expression can be simplified down to $\frac{17}{4}x$, upon which computation gives $17 \cdot 2 = \boxed{34}$.
4. To use the least number of coins, we should use as many large-valued coins as possible. Thus, to make 37 cents, we would use 1 quarter, 1 dime, and 2 pennies, for a total of $\boxed{4}$ coins.
5. Mr. Tree's height must increase by $100 - 87 = 13$ feet every day. Therefore, tomorrow Mr. Tree will be $100 + 13 = \boxed{113}$ feet tall.
6. A dollar in pennies is 100 pennies, while a dollar in dimes is 10 dimes. Since 2 dimes weigh the same as 1 penny, 10 dimes weigh the same as 5 pennies. Then the weight of 100 pennies is $\boxed{20}$ times heavier than the weight of 5 pennies.
7. A yard is 3 feet, so two furlongs are $2 \times 220 \times 3$ feet. This, when converted to Ryans is $\frac{2 \times 220 \times 3}{5} = 88 \times 3 = \boxed{264}$
8. The worst possible situation is that I ran in the opposite direction as my ducks, in which case the distance between us would be $50 + 27 = \boxed{77}$ meters.
9. Writing out 7!, we have $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7$. Writing out the prime factorizations, we have $7! = 1 \cdot 2 \cdot 3 \cdot 2^2 \cdot 5 \cdot (2 \cdot 3) \cdot 7 = 2^4 \cdot 3^2 \cdot 5^1 \cdot 7^1$. So $a = 4, b = 2, c = 1$, and $d = 1$, so $a - b + c - d = 4 - 2 + 1 - 1 = \boxed{2}$.
10. We write out the first 15 terms. 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610. We note that the last number is even. Each even number is the sum of the previous 2 odd numbers, so when calculating $x - y$, all even terms cancel out with the previous 2 odd terms (for example, $2 - 1 - 1 = 0$ and $8 - 5 - 3 = 0$). So $x - y = \boxed{0}$. (Note that the observation that the last term is even was sufficient; we didn't have to calculate all 15 terms.)
11. The largest value of $a - b$ can be attained by choosing the greatest value of a and the least value of b. This gives $a = 15, b = -3$, for a difference of 18. Similarly, $a - b$ is minimized by minimizing a and maximizing b: $a = 10$, $b = 7$, for a difference of 3. So, the sum of the smallest and largest values of $a - b$ is $18 + 3 = \boxed{21}$
12. There are 21 choices for the first consonant. There are 5 choices for the first vowel. There are 20 choices for the second consonant. There are 4 choices for the second vowel. In total, there are $21 \cdot 5 \cdot 20 \cdot 4 = \boxed{8400}$ possible code words.
13. For every possibility that there are more heads than tails, there is a symmetric case where there are more tails than heads by just swapping all the heads to tails and tails to heads. Thus, the desired probability is $\frac{1}{2}$, yielding an answer of $1 + 2 = \boxed{3}$.
14. Notice that if he saw two distinct integers, he could not have concluded this, since both the absolute difference and sum of these two integers could have been on the third piece of paper. Therefore, he must have seen the same two integers. In addition, since the difference when an integer is subtracted from itself is 0, 46 must be the sum of the two integers. Therefore, the integer on the first piece of paper is $\frac{46}{2} = \boxed{23}$.

15.

$$\begin{aligned} a^3+b^3 &= (a+b)(a^2-ab+b^2) \\ &= 7[(a+b)^2-3ab] \\ &= 7[7^2-3(13)] \\ &= 7(49-39) \\ &= \boxed{70}. \end{aligned}$$

16. If O_1 and O_2 are the centers of the circles, and the circles intersect at A and B, then $\triangle O_1AO_2$ and $\triangle O_1BO_2$ are equilateral. Thus, $\angle AO_1B = \angle AO_2B = 120°$. So the perimeter of the region of points covered by at least one of the two circles consists of two-thirds the perimeter of one circle and two-thirds the perimeter of the other circle. The total perimeter is therefore four-thirds the perimeter of one circle, or $\frac{4}{3} \times 2\pi(\frac{12}{\pi}) = \boxed{32}$.

17. This question is essentially asking for the surface area of the pancakes. The area of the top face of the pancakes is $6^2 \times \pi$ because each smaller radius pancake is within the boundaries of the 6 inch one. Now, looking at the side of the pancake, the area is $2r\pi \times 1 = 2r\pi$ where r is the radius of the pancake. In total, the side area is $2\times2\times\pi+2\times4\times\pi+2\times6\times\pi = 24\pi$. In total, the surface area is $36\pi+24\pi = 60\pi$. Therefore, the answer is $\boxed{60}$.

18. Rewrite $2xy+7x-8y=0$ as follows:

$$(x-4)(2y+7)+28=0 \implies (x-4)(2y+7)=-28$$

Since both $x-4$ and $2y+7$ are always integers, $(x-4,\ 2y+7)$ must be a factor pair of -28. Furthermore, notice that $2y+7$ is always odd. So, the possible factor pairs are $(-28,1)$, $(-4,7)$, $(4,-7)$, and $(28,1)$. These correspond to $x=-24,0,8,$ and 32, respectively. So, the sum of all integers x is $\boxed{16}$.

19. Notice that because the hexagon is equiangular, each angle must be 120 degrees. Now, we can inscribe this hexagon in a rectangle, with the opposite sides of 2 and 4 lying on opposite sides of the rectangle. To find the area of the hexagon, we can subtract the area of the four corner triangles, which are $30-60-90$ triangles because of the measure of each angle in the hexagon. The triangles with hypotenuse 4 will have area $2\sqrt{3}$ each, and the triangles with hypotenuse 2 have area $\frac{\sqrt{3}}{2}$ each. The total hexagon will have an area of $(2\sqrt{3}+\sqrt{3})(1+4+1) = 18\sqrt{3}$. Subtracting the triangle areas, we obtain $18\sqrt{3}-4\sqrt{3}-\sqrt{3} = 13\sqrt{3}$, giving an answer of $13+3=\boxed{16}$.

20. We figure out what digits work, and then calculate how many ways there are to rearrange those digits. We note that in the example given, 5255525, there are two 2's and five 5's. So considering other possibilities, there are seven 7's, one 1 and six 6's, two 2's and five 5's, three 3's and four 4's, and finally, one 1, one 2, and one 4. For each case, we can reorder the digits as needed.
For 7777777, there's 1 ordering.
For 1666666, there's $\binom{7}{1} = 7$ orderings.
For 2255555, there's $\binom{7}{2} = 21$ orderings.
For 3334444, there's $\binom{7}{3} = 35$ orderings.
For 1224444, there's $\frac{7!}{4!2!} = 105$ orderings.
In total, there are $1+7+21+35+105 = \boxed{169}$ numbers.

21. The circumference of the circle in kilometers is $2\pi \times 6 = 12\pi$. Masha walked 8π kilometers, which is $\frac{2}{3}$ of the circumference. Masha's path back home is one side of an equilateral triangle that is inscribed in the circular park. The side length of this triangle is $6 \times \frac{3}{2} \times \frac{2}{\sqrt{3}} = 6\sqrt{3}$ kilometers. $6+3=\boxed{9}$.

22. Let A be the point $(0,0)$ and B be the point $(6,8)$. Using the distance formula, we know that $AB = \sqrt{6^2+8^2} = \sqrt{100} = 10$.

Now, let's consider the orientation of the square. We have two cases to consider.

Case 1: A and B are adjacent vertices of the square. Then, AB is the side length. The area of the square is then simply the square of the side length, or $10^2 = 100$ square units.

Case 2: A and B are diagonally opposite vertices of the square. In a square, the diagonals have the same length, so the length of the diagonal of the square is 10 units. The side length of the square is then the diagonal divided by $\sqrt{2}$ (as the diagonal, side, and a perpendicular from one vertex to the diagonal form a 45-45-90 triangle). Thus, the side length is $\frac{10}{\sqrt{2}} = \frac{10\sqrt{2}}{2} = 5\sqrt{2}$ units, and the area of the square is $(5\sqrt{2})^2 = 50$ square units.

So, the difference between the maximum and minimum possible area of the square is $100 - 50 = \boxed{50}$ square units.

23. We can start by adding $f(x)$ and $g(x)$ to get $-2x^2 + 8x + 42$. Rearrange this into vertex form: $-2 \cdot (x^2 - 4x) + 42 = -2 \cdot (x-2)^2 + 50$. The vertex of a parabola is located at $x = 2$, and since this is a downward facing parabola, this x will give the maximum value. Plugging in 2 gives $-2 \cdot 2^2 + 8 \cdot 2 + 42 = \boxed{50}$.

24. First note that we cannot have 0 digit to reach the front of the number during the pushing. Let us casework based on the number of digits.

    (a) 1 digit: all 1-digit numbers will return to itself. This gives us 9 solutions.

    (b) 2 digits: Pushing ab three times will result in ba if a and b are non-zero. Then, $a = b$ for ab to return to itself. This gives us 9 solutions.

    (c) 3 digits: Pushing abc three times will result in abc for any non-zero a, b, c. This gives us $9^3 = 729$.

    (d) 4 digits: Pushing $abcd$ three times will result int $bcda$ for non-zero a, d, c. We must have $a = b = c = d$, giving us 9 solutions.

    (e) 5 digits: Pushing $abcde$ three times will result in $cdeab$. We must have non-zero c, b, a and $a = b = c = d = e$. This gives us 9 solutions.

    (f) 6 digits: Pushing $abcdef$ three times will give us $defabc$ for non-zero a, e, f and $a = d, b = e, c = f$. This gives us $9^3 = 729$ solutions.

    In total, we have $4 \cdot 9 + 2 \cdot 729 = \boxed{1494}$ solutions.

25. It can be seen by expressing the numbers in the set in binary that all decimal numbers up to $2^0 + 2^1 + \cdots + 2^6 = 2^7 - 1 = 127$ can be expressed in the desired form. Thus, there are $\frac{127+1}{2} = \boxed{64}$ odd numbers that can be expressed.

26. The horizontal line intersects the vertex of the parabola. Therefore, $b = 4$. The x-coordinates of the other two intersection points can be solved with $b = ax^2 - b$ or $x = \pm\sqrt{\frac{2b}{a}} = \pm\frac{2\sqrt{2}}{\sqrt{a}}$. The distance between these two points is $\frac{4\sqrt{2}}{\sqrt{a}} = 4$ so $a = \boxed{2}$.

27. Upon drawing the diagram, we draw a perpendicular from point A to BQ. Let the point of intersection be D, so that $AD \perp BQ$, as shown in the diagram. Now, since $ADQP$ is a rectangle due to all angles being 90° and $AP = 4$, $DQ = 4$. Then $BD = BQ - DQ = 9 - 4 = 5$. AB is the sum of the two radii, so $AB = 4 + 9 = 13$. By Pythagorean Theorem on $\triangle ABD$, since $AB = 13$ and $BD = 5$, we know $AD = 12$. Thus, $PQ = 12$ as well. By AA similarity, since $\angle C$ is a shared angle and $\angle APC = \angle BQC = 90$, $\triangle ACP\ \triangle BCQ$. We can then form the ratios $\frac{CP}{AP} = \frac{CQ}{BQ}$, so $\frac{CP}{4} = \frac{CP+12}{9}$. Solving this equation, we obtain $CP = \frac{48}{5}$. To find the area of $\triangle ACP$, we do $CP \cdot \frac{AP}{2}$, which is $\frac{48}{5} \cdot \frac{4}{2} = \frac{96}{5}.\frac{a}{b} = \frac{96}{5}$, so $a + b = \boxed{101}$.

28. Assume that stopwatch A is displaying digits $X{:}YZ$ when we stop it. The number of seconds that elapsed is going to be $60X + 10Y + Z$. Taking this (mod 9), we have the value $6X + Y + Z$.

    However, as the digits can be rearranged to give the digits of B, B is congruent to $X + Y + Z$ (mod 9). Since the quantities $6X + Y + Z$ and $X + Y + Z$ are both 0 (mod 9), their difference will also be 0 (mod 9). So, $(6X + Y + Z) - (X + Y + Z) = 5X$. Therefore, $5X$ must be divisible by 9, so X is also divisible by 9.

Taking the least possible value of X, 9, we have that the total number of seconds elapsed, the reading on B, is between 540 and 599, and the reading on A begins with a 9. Additionally, the rightmost digit will be the same for both A and B, as $60X + 10Y + Z$ is congruent to $Z \pmod{10}$.

Therefore, the only possible times in this case are of the form $59Z$ seconds, displaying $9 : 5Z$ on stopwatch A, with the smallest possible value being $\boxed{590}$.

29. To maximize the fraction, we want to maximize the numerator and minimize the denominator (While making it negative). So, set $a = 9$. After that, we want $b - c = 1$, while also maximizing b. So, set $b = 8, c = 9$. Notice that if $b = 9$, then the denominator is guaranteed to be positive (it can't be 0), making the maximum less than 9. So, the maximum occurs when $a = 9, b = 8, c = 9$, so the answer is $\boxed{989}$

30. All square numbers will have a units digit as one of $\{1, 4, 5, 6, 9\}$. As we already know 1 is perplexing, and no other one-digit squares have a 1 in the units digit, we need to find the first two 4-digit squares which have a 4 in the units digit. 1024 is perplexing, as $32 * 32 = 1024$, and 1444 is perplexing, as $38 * 38 = 1444$. Therefore, our answer is $1 + 1024 + 1444 = \boxed{2469}$.

31. Notice that the number $555...555_6$ with 2023 occurrences of the digit 5 is equal to $6^{2023} - 1$ in base 10. In addition, notice that $6^{2023} = 6 \cdot (6^2)^{1011} = 6000...000_{36}$, where there are 1011 occurrences of the digit 0. If we let D denote the digit for 35 in base 36, we then have that $6^{2023} - 1 = 5DDD...DDD_{36}$, where there are 1011 occurrences of the digit D. We want the digit sum of this base 36 integer, which is $5 + 1011 \cdot 35 = \boxed{35390}$.

32. We can realize that if the least common multiple is contained in the subset, then the only other elements of the subset would be divisors of the least common multiple. For example, if we take the number 12, its divisors are $1, 2, 3, 4, 6, 12$. 12 must be in the subset, but the remaining 5 numbers can be in it or not, so there are 2^5 of such subsets for the largest element being 12. For all the least common multiples that are primes, there would be 2 ways to create this subset. Since there are 7 primes from 1 to 17, these comprise of $2 \cdot 7 = 14$ of the subsets. For the remaining subsets, there would be $1+2^2+2^3+2^3+2^2+2^3+2^5+2^3+2^3+2^4 = 97$. When added to 14, this would be $97 + 14 = \boxed{111}$.

33. By Vieta's, $r + s + t = -22$, $rs + rt + st = -14$, $rst = -2$. $\frac{1}{r} + \frac{1}{s} + \frac{1}{t} = \frac{st + rt + rs}{rst} = \frac{-14}{-2} = 7$. $\frac{1}{rs} + \frac{1}{rt} + \frac{1}{st} = \frac{r + s + t}{rst} = \frac{-22}{-2} = 11$. So the answer is $7 \cdot 11 = \boxed{77}$.

34. Extend lines $\overline{BC}$ and $\overline{AD}$ to meet at E. Since $\angle C = \angle D = 45^\circ$, it follows that $\triangle CED$ is an isosceles right triangle with hypotenuse $CD = 14\sqrt{2}$. Thus $EC = ED = 14$, so $EB = EC - BC = 14 - 6 = 8$. Since $\triangle AEB$ is a right triangle, by the Pythagorean Theorem, it follows that $EA = \sqrt{10^2 - 8^2} = 6$. Thus, the area of quadrilateral $ABCD$ is the area of $\triangle CED$ minus the area of $\triangle AEB$, which is $\frac{1}{2}(14\times14) - \frac{1}{2}(6\times8) = \boxed{74}$.

35. First we note that when placing a plus sign, the value of the expression (mod 9) is invariant. This is because the sum of the digits of a number has the same remainder when divided by 9 as the number itself. Therefore, n must be congruent to 6 (mod 9).

    6 and 15 can be achieved with one 111 and either one 11 and one 1 or 12 ones. 24 can be achieved with 11 elevens and 2 ones. All other numbers congruent to 6 (mod 9) between 33 and 123 can be achieved by exchanging an 11 for 11 ones.

    In total, all of the numbers congruent to 6 (mod 9) between 6 and 123, inclusive, can be reached, giving $\boxed{14}$ in total.

36. Notice that E being a divisor of BEE, EBE, or EEB is equivalent to E being a divisor of B00, B0, or B, respectively. Thus, E is a divisor of B00 but not B0, implying that either E equals 4 and B is odd, or E equals 8 and B is either 2 or 6. So the only options are:

    $$\text{BEE} = 144,\ 344,\ 544,\ 744,\ 944,\ 288,\ 688.$$

    The product of the digits of these numbers, respectively, are:

    $$(144, 16),\ (344, 48),\ (544, 80),\ (744, 102),\ (944, 144),\ (288, 128),\ (688, 384).$$

    The only product of digits that matches "SET" is the one corresponding to the secret code being $\boxed{688}$.

37. We can do this question by recursion. Let the number of sequences satisfying this with length n be R_n. Consider the last digit in the sequence. It can be either 1, 4, or 7, which would yield $3R_{n-1}$ of such sequences. However, we must subtract those that have a 17 in the last two positions, and there are R_{n-2} of these. This would mean our recursion is $R_n = 3R_{n-1} - R_{n-2}$. Then, we can find R_1 and R_2 to solve for the rest of the values. When there is only 1 number in the sequence, we can never get 17, so $R_1 = 3$. In a two digit sequence, there is only one way to have a 17, so $R_2 = 3^2 - 1 = 8$. If we continue with this recursion, we obtain that $R_3 = 21, R_4 = 55, R_5 = 144, R_6 = 377, R_7 = 987$. Therefore, there are $\boxed{987}$ of such sequences.

38. Suppose chord $\overline{AB}$ intersects sides $\overline{PQ}$ and $\overline{QR}$ of square $PQRS$ at X and Y, respectively. By symmetry, chord $\overline{AB}$ should be parallel to diagonal $\overline{PR}$ of the square. In particular, since $\triangle PQR$ is an isosceles right triangle, $\triangle QXY$ should also be an isosceles right triangle.

    Since $XY = 6$, it follows that $QX = QY = 3\sqrt{2}$. By Power of a Point, $(AX)(XB) = (PX)(XQ)$, so $(6)(12) = (PX)(3\sqrt{2})$, so $PX = 12\sqrt{2}$. Therefore, the side length of the square is $15\sqrt{2}$ and its area is $\boxed{450}$.

39. Taking the diagonal cross section of the cube, we get a rectangle with length $2\sqrt{2}$ and width 2. The circle sits in the middle tangent to the top and the bottom. Setting the length outside the circle to x. We use Power of a Point to get $2 = x \cdot \sqrt{6}$. We then get a ratio of $\frac{\frac{2}{\sqrt{6}}}{\sqrt{6}} = \frac{1}{3}$ giving an answer of $\boxed{4}$.

40. If there are n working batteries, the number of working pairs is going to be $\binom{n}{2}$. After 95 battery pairs are tested, we are uncertain about the remaining ten pairs, so the range of possible values for the running total is going to be $[\binom{n}{2} - 10, \binom{n}{2}]$. The lower bound is achieved when all of the remaining battery pairs are working, and the upper bound is achieved when all of the remaining battery pairs are not.

    For there to be any uncertainty about the number of working batteries, the ranges for 2 or more values of n must overlap, and the running total will need to be in the range of intersection. Therefore, we have that $\binom{n}{2} - 10 \leq \binom{n-1}{2}$ for some n. Expanding we get $n \cdot (n-1) - 20 \leq (n-1) \cdot (n-2)$, so $2 \cdot (n-1) \leq 20$ and $n \leq 11$.

    Taking the greatest possible value of n, we see that the intersection only has one value in it, $\binom{11}{2} - 10 = \binom{10}{2} = \boxed{45}$.