# Joe Holbrook Memorial Math Competition

5th Grade

October 20, 2024

1. Observe that $\spadesuit + \spadesuit + \spadesuit = 3 \times \spadesuit = 27$, which implies $\spadesuit = 27 \div 3 = \boxed{9}$.
2. Consider trying to add bottles until adding another bottle would overflow the bucket. This is like dividing 5000 by 473, and we can see that the most number of times that 473 goes into 5000 is 10, as 10 * 473 is 4730 and adding 473 to that would push us over 5000. Therefore, there can be a maximum of $\boxed{10}$ bottles of water.
3. Their difference is $(2024 + 2023) - (2024 - 2023) = 2024 + 2023 - 2024 + 2023 = 2023 + 2023 = \boxed{4046}$.
4. $(2 + 0 + 2 + 4) \times (20 + 24) = 8 \times 44 = \boxed{352}$.
5. $24 \times 25 + 25 \times 26 = 25 \times (24 + 26) = 25 \times 50 = \boxed{1250}$.
6. We work backwards. We start with having two slices, and we know a third was eaten meaning there must have been three slices. We know half was eaten, meaning there must have been $\boxed{6}$ slices in total.
7. The prime digits are $2, 3, 5,$ and $7$. Clearly the units digit cannot be 2 or 5, so there are only 8 numbers to check: $23, 27 = 9 \times 3, 33 = 11 \times 3, 37, 53, 57 = 19 \times 3, 73, 77 = 7 \times 11$, meaning there are $\boxed{4}$ numbers which are prime in that list.
8. Since 50 students completely fill 2 buses, a single bus must have a maximum capacity of $50 \div 2 = 25$ students. So 289 students would need at least $289 \div 25 = 11.56$ buses, or $\boxed{12}$ buses.
9. If Tony the Tree was 26 feet tall yesterday, it means Tony the Tree will be $26 \times 2 = 52$ feet tall today, and tomorrow he will be $52 \times 2 = \boxed{104}$ feet tall.
10. If Jeremy randomly selects all of the consonants before any vowels, he will pick 20 letters. His next letter, the 21st letter, must necessarily be a vowel. Because this is the worst possible scenario, Jeremy must select $\boxed{21}$ random letters to ensure he picks at least one vowel.
11. Every 3 in 5 ducks in a family are baby ducks, so there are $\dfrac{3}{5} \times 120 = \boxed{72}$ baby ducks.
12. We know 20 feet equals $20 \times 12$ inches. So, $\dfrac{20 \times 12}{10}$ is the number of purlunkles you need, or $\boxed{24}$.
13. Arnav has a $\dfrac{1}{10}$ probability of hitting a bullseye on each shot. Since the shots are independent, the probabilities multiply. The probability of getting two bullseyes in a row is $\dfrac{1}{10} \times \dfrac{1}{10} = \dfrac{1}{100}$, or $\boxed{101}$ as desired.
14. Using the Pythagorean theorem, Amy must walk $\sqrt{12^2 + 16^2} = 20$ feet. Sabrina walks a total of $12 + 16 = 28$ feet, so the difference is $28 - 20 = \boxed{8}$.
15. Because there are 80 members, $80 \cdot \frac{90}{100} = 72$ people eat one donut, $80 \cdot \frac{5}{100} = 4$ eat two donuts, and $80 \cdot \frac{5}{100} = 4$ eat zero donuts. In total, they ate $72 \cdot 1 + 4 \cdot 2 + 4 \cdot 0 = 80$ donuts. Thus, our average is $80/80 = \boxed{1}$ donut.
16. Consider the worst case, in which he picks distinct pairs of socks right until his last pick. This would mean that for 7 picks in a row he needs distinct socks, and therefore $\boxed{7}$ pairs of distinct socks. Notice that on his 8th pick it does not matter what sock he picks, as whatever he picks it will be a pair with another one already chosen. This situation can also be modeled with 7 pigeonholes and 8 pigeons.
17. Increasing the length multiplies the area by $\frac{5}{4}$, so we must multiply the width by $\frac{8}{5}$ to double the area, since $\frac{5}{4} \cdot \frac{8}{5} = 2$. $\frac{8}{5}$ equals 160 percent, so the answer is $\boxed{60}$.

18. If Gabriele's squares have an area of 4, then the side length of Gabriele's square is 2, as $2 \times 2 = 4$. Then the perimeter of his square is 8. Note that no matter how many squares Gabriele draws, as long as they're all the same, the total perimeter will always be double the total area. Since the total area is 100, the total perimeter will be double that, or $\boxed{200}$.

19. We can first see that Danny's new flask after the theft has $200 + 400 = 600$ mL of liquid. Since $\frac{1}{2}$ of this liquid is coffee, there will be $600 \times \frac{1}{2} = 300$ mL of coffee in Danny's flask after mixing. Danny's flask originally had $400 \times \frac{3}{5} = 240$ mL of coffee. Therefore, $300 - 240 = 60$ mL of coffee was gained from Nikhil's flask. The percentage of coffee in Nikhil's flask is $\frac{60}{200} \times 100 = \boxed{30\%}$

20. Drawing it out, this is a right triangle with legs of length 3 and 4, so we can use the Pythagorean theorem to find the third side:
$$\sqrt{3^2 + 4^2} = \sqrt{25} = \boxed{5}.$$

21. If Caleb has a dollar of pennies and a dollar of dimes, he has exactly 100 pennies and 10 dimes, which adds to 110 coins. If he loses half of these, he will lose 55 coins. Assuming he lost the least amount of money possible, he will only lose pennies, meaning he loses 55 cents and keeps $\boxed{145}$.

22. Every visit, the day of the week advances by 3 days, and we need to move 1 day ahead of Saturday. This occurs after 5 visits, because $5 \cdot 3 = 15$ leaves a remainder of 1 when divided by 7. Thus, after $\boxed{85}$ days, the day will be Sunday.

23. We can analyze $2024 \pmod{100}$. Since $4|2024$, only $2024 \pmod{25} \equiv -1 \pmod{25}$ matters. We conclude that the final number will be $1 \pmod{25}$ and $0 \pmod{4}$, so the answer must be $\boxed{76}$.

24. Notice that each of the 6 pairs of numbers that are formed by taking 2 distinct values from $1, 2, 3, 4$ are either two adjacent numbers in a row, two adjacent numbers in a column, or diagonally opposite numbers. Therefore, we have that the sum is equal to
$$1 \cdot 2 + 1 \cdot 3 + 1 \cdot 4 + 2 \cdot 3 + 2 \cdot 4 + 3 \cdot 4 = 35$$
and subtracting out the sum of the product of the values on each diagonal.

    The sum of the product of the values on each diagonal is either $1 \cdot 2 + 3 \cdot 4 = 14$, $1 \cdot 3 + 2 \cdot 4 = 11$ or $1 \cdot 4 + 2 \cdot 3 = 10$. To maximize the sum of the product of the values in each row and each column, we need to minimize the sum of the product of the values each diagonal. Therefore, our desired answer is $35 - 10 = \boxed{25}$.

25. Let $a$ be an angle of the triangle. The other 2 angles can be expressed as $3a$ and $9a$. Since we know that the sum of the angles of a triangle is 180 degrees, we can write the equation $a + 3a + 9a = 13a = 180$, $a = \frac{180}{13}$. The closest integer to this value is $\boxed{13}$.

26. Since we want the largest number, the side length of all these polygons must be 1. The least number of side lengths a polygon can have is 3, and this has perimeter 3. If we continue this way, we find that we want for $3 + 4 + \ldots + n < 400$. This becomes $(n+3)\left(\frac{n-2}{2}\right) < 400$, or we want $(n+3)(n-2) < 800$. The largest value of $n$ here is 27, so the number of polygons he made was $27 - 3 + 1 = \boxed{25}$.

27. Let $r$, $b$, and $g$ be the number of red, blue, and green crayons, respectively. Then, we can set up the following equations:

    (a) $b + g = 7$
    (b) $r + g = 3$
    (c) $r + b = 8$

    Adding up all the equations, we get $2r + 2b + 2g = 18$. Therefore, the total number of crayons, $r + b + g$, equals $\boxed{9}$.

28. Note that if $A$ and $C$ have a last digit of $A$, $C$ must be 0. Similarly, we can see that $C$ and $B$ also sum to $B$. Now, the only value for the first two digits of $AC$ is 10, since there are no carries in the first two columns of addition. Thus, we just want to maximize the last two digits of ACBA, while keeping B + A equal to 10. This happens when B is 9 and $A$ is 1. Our final answer is $\boxed{1091}$.

29. Let $h$ and $p$ be the number of hexagons and pentagons, respectively. We need $6h = 5p$, which means we need to find the least common multiple of 6 and 5. This is 30, so we want 5 hexagons and 6 pentagons to satisfy this equation with the least number of shapes. Our answer is thus $5 + 6 = \boxed{11}$.

30. Notice we can write each of these five integers as integer multiples of the fifth number as it has the smallest magnitude. Thus, let the fifth number be $x$. The first number is $2x$, the second number is $6x$, the third number is $30x$, and the fourth number is $120x$. Notice the pairwise differences of these numbers are integer multiples of $x$. The pairwise difference we want is prime so it must be $x$ as there are no differences of $37x$. Thus, $x = 37$. The sum of all the numbers is then

$$x + 2x + 6x + 30x + 120x = 159x = \boxed{5883}.$$

31. If $3^{\frac{x}{2}} = 2$, then squaring gives $3^x = 2^2 = 4$. Furthermore, we can square two more times to get $9^x = (3^x)^2 = 4^2 = 16$ and $9^{2x} = (9^x)^2 = 16^2 = 256$. Finally, we multiply by 9 once to get $9^{2x+1} = 9^{2x} \cdot 9^1 = 256 \cdot 9 = \boxed{2304}$.

32. The average of $y$, $y+23$, and $y+46$ is equal to $\dfrac{(y) + (y+23) + (y+46)}{3} = \dfrac{3(y+23)}{3} = y + 23$. Since this average is the answer to the question, it is equal to $x$. Then, since y is half of x, $y = \dfrac{x}{2}$, and $\dfrac{x}{2} + 23 = x$. Simplifying, we find that $x = \boxed{46}$.

33. Note that at $x = 5$ and above, $x!$ will be a multiple of 10 and thus $(x!)^2$ will be a multiple of 100. Thus, the remainder when the infinite sum is divided by 100 is the same as the remainder when $(1!)^2 + (2!)^2 + (3!)^2 + (4!)^2$ is divided by 100. $(1!)^2 + (2!)^2 + (3!)^2 + (4!)^2 = 1^2 + 2^2 + 6^2 + 24^2 = 1 + 4 + 36 + 576 = 617$, which has a remainder of $\boxed{17}$ when divided by 100.

34. Since the volume of the balloon multiplied by 8, the surface area multiplied by $8^{\frac{2}{3}}$, or 4 times, and thus a new surface area of $\boxed{128}$ square inches.

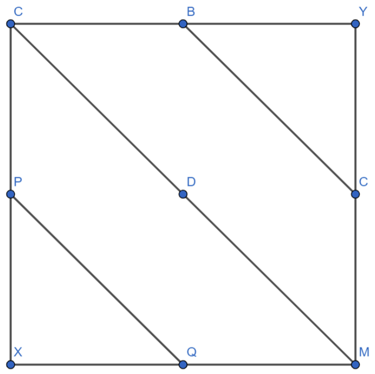
35. Note that every time a deer passes by, a rabbit passes by, so the dog barks anyways. Thus, we can ignore the deer. If cats pass by every 3 minutes, in an hour timeframe, they pass by $60/3 = 20$ times. Rabbits pass by $60/5 = 12$ times. However, we overcount for when the dog barks only once for when cats and rabbits pass at the same time. They do this every 15 minutes, so $60/15 = 4$ times total. Thus, the dog barks $20 + 12 - 4 = \boxed{28}$ times total in the hour timeframe.

36. If some person takes the red marble, there are only 3 people who therefore cannot have taken the blue marble: the person who took the red marble and the two people sitting next to them. Therefore, for each of the 26 people who could take the red marble, 23 others could have taken the blue marble. Every other person will have a green marble. Thus, there are $26 * 23 = \boxed{598}$ distinct ways to distribute the marbles.

37. The expression can be rewritten as $(x + y)^2 + (y + 2)^2 - 4$. By the Trivial Inequality, the minimum $0 + 0 - 2 = -4$ occurs when $x + y = 0$ and $y + 2 = 0$ to make the squared terms equal to their minimum value of 0, giving $(x, y) = (2, -2)$. Therefore, we have $a = 2$ and $b = -2$, giving an answer of $-4 \cdot 2 \cdot -2 = \boxed{16}$.

38. If Cam picks a number higher than 6, he will win automatically, as the highest number Devin can pick is 6. The probability of this is $\dfrac{4}{10}$. For all other numbers, if Devin and Cam don't tie, the chance Cam wins is $\dfrac{1}{2}$. The probability that Devin rolls Cam's number is $\dfrac{1}{6}$, so the probability they don't tie is $\dfrac{5}{6}$. Then, if Devin and Cam do tie, the probability that Cam wins is the same as the initial probability. So, let $S$ be the probability of Cam winning, then $S = \dfrac{4}{10} + \dfrac{6}{10} \times \dfrac{5}{6} \times \dfrac{1}{2} + \dfrac{6}{10} \times \dfrac{1}{6} \times S$. $S = \dfrac{4}{10} + \dfrac{1}{4} + \dfrac{1}{10} \times S$. $\dfrac{9}{10} \times S = \dfrac{13}{20}$. So, $S = \dfrac{13}{18}$, which gives an answer of $13 + 18 = \boxed{31}$.

39. Drop the altitude from $A$ to $D$ on $CM$. The area of $CAM$ is $\dfrac{CM \times AD}{2}$. As $CM = 4\sqrt{2}$, we need $AD \geq \sqrt{2}$.




So, draw the lines that are $\sqrt{2}$ away from $CM$. As the distance from $CM$ to $X$ and $Y$ is $2\sqrt{2}$, the distance from $PQ$ to $X$ is $\sqrt{2}$ and the distance from $BC$ to $Y$ is $\sqrt{2}$. As $PQX$ and $BCY$ are both similar to $CMY$, we know their sides have a ratio of $\frac{1}{2}$, meaning their areas have a ratio of $\frac{1}{4}$. As $CMX$ and $CMY$ are each $\frac{1}{2}$ the square, the area of $PQX$ and $BCY$ is both $\frac{1}{8}$ of the entirety of the entire square, for a combined $\frac{1}{4} \rightarrow \boxed{5}$.

40. Parity argument gives that one of the primes, WLOG $r$, is 2, since 617 is odd and the sum of four odd numbers is even. Therefore, $pq + 2q + 2p + 2pq = 617$, so $3pq + 2q + 2p = 617$, so $9pq + 6q + 6p = 1851$, and $9pq + 6q + 6p + 4 = 1855$. Thus $(3p + 2)(3q + 2) = 1855 = 5 \cdot 7 \cdot 53$, so the only possibility is $p$ and $q$ are 17 and 11. Our answer is $2 + 11 + 17 = \boxed{30}$.