Joe Holbrook Memorial Math Competition

6th Grade

October 20, 2024

1. Observe that $\spadesuit + \spadesuit + \spadesuit = 3 \times \spadesuit = 27$, which implies $\spadesuit = 27 \div 3 = \boxed{9}$.

2. Each pizza is half a foot thick, or 0.5 feet thick. Dividing the height we need with the height of each pizza, we get $1454 \div 0.5 = \boxed{2908}$ pizzas.

3. If Tony the Tree was 26 feet tall yesterday, it means Tony the Tree will be $26 \times 2 = 52$ feet tall today, and tomorrow he will be $52 \times 2 = \boxed{104}$ feet tall.

4. $(2 + 0 + 2 + 4) \times (20 + 24) = 8 \times 44 = \boxed{352}$.

5. The least positive multiple of 11 is 11, and the greatest positive multiple of 11 less than 200 is 198 (which is equal to 11×18). Thus, the total amount of multiples satisfying this question is $\boxed{18}$.

6. He buys $10 \times 90 = 900$ sticky notes in total. He uses $3 \times 180 = 540$ of them. Therefore, he has $900 - 540 = \boxed{360}$ remaining.

7. If Jeremy randomly selects all of the consonants before any vowels, he will pick 20 letters. His next letter, the 21st letter, must necessarily be a vowel. Because this is the worst possible scenario, Jeremy must select $\boxed{21}$ random letters to ensure he picks at least one vowel.

8. Arnav has a $\dfrac{1}{10}$ probability of hitting a bullseye on each shot. Since the shots are independent, the probabilities multiply. The probability of getting two bullseyes in a row is $\dfrac{1}{10} \times \dfrac{1}{10} = \dfrac{1}{100}$, or $\boxed{101}$ as desired.

9. We know 20 feet equals 20×12 inches. So, $\dfrac{20 \times 12}{10}$ is the number of purlunkles you need, or $\boxed{24}$.

10. If Gabriele's squares have an area of 4, then the side length of Gabriele's square is 2, as $2 \times 2 = 4$. Then the perimeter of his square is 8. Note that no matter how many squares Gabriele draws, as long as they're all the same, the total perimeter will always be double the total area. Since the total area is 100, the total perimeter will be double that, or $\boxed{200}$.

11. The coins used would be 1 quarter, 2 dimes, and 4 pennies, for $\boxed{7}$ coins.

12. We can start by finding that the perimeter of the rectangle in terms of s will also be $2 \times (7s + 3s) = 20s$. This means that $20s = 100$, giving us $s = 5$. We can then find the area of the square to be $s^2 = 5^2 = \boxed{25}$.

13. Every visit, the day of the week advances by 3 days, and we need to move 1 day ahead of Saturday. This occurs after 5 visits, because $5 \cdot 3 = 15$ leaves a remainder of 1 when divided by 7. Thus, after $\boxed{85}$ days, the day will be Sunday.

14. $4\Delta 5 = 4 \times (4 + 5) = 36$

    $3\Delta 36 = 3 \times (3 + 36) = \boxed{117}$

15. Let a be the number of 9-scoop ice cream cones Bear buys. For Bear to buy the least amount of cones possible, we want to maximize how many 9-scoop ice cream cones he buys. In other words, we need to find the largest a so that $100 - 9a$ is divisible by 7. Note that $a < 12$, so the only possible values of a are $a = 1$ and $a = 8$, and since $8 > 1$, $a = 8$. Then $100 - 9a = 100 - 9(8) = 28 = 7 \times 4$, so Bear buys 8 9-scoop cones and 4 7-scoop cones and our answer is $8 + 4 = \boxed{12}$.

16. Increasing the length multiplies the area by $\frac{5}{4}$, so we must multiply the width by $\frac{8}{5}$ to double the area, since $\frac{5}{4} \cdot \frac{8}{5} = 2$. $\frac{8}{5}$ equals 160 percent, so the answer is $\boxed{60}$.

17. If Caleb has a dollar of pennies and a dollar of dimes, he has exactly 100 pennies and 10 dimes, which adds to 110 coins. If he loses half of these, he will lose 55 coins. Assuming he lost the least amount of money possible, he will only lose pennies, meaning he loses 55 cents and keeps $\boxed{145}$.

18. The only two digit palindromes that becomes a three digit number when 13 is added to it are 88 or 99, and testing shows that only $x = \boxed{88}$ works.

19. There are 8 prime numbers less than 20, and the only even prime number is 2. So, the probability is $\frac{1}{8}$, and the answer is $1 + 8 = \boxed{9}$.

20. $f(14) = 17$, $f(17) = 50$, thus $f(f(f(14)))) = f(50) = \boxed{25}$.

21. We can break this problem into cases based on how many houses Dolly builds. If she builds 1 house, there are 6 different spots where she could build it. If she builds 2 houses, she can start by placing 1 house on one of the 6 vertices, and since she can't place a house on either of the adjacent vertices, she has 3 choices for the second house. $6 \cdot 3 = 18$ choices for placing these houses, but each choice is counted twice, so there are actually $\frac{18}{2} = 9$ ways of placing 2 houses. If there are 3 houses, Dolly must places them on alternating vertices, giving 2 ways of placing 3 houses, and it is apparent that trying to place more than 3 houses would result in multiple houses being adjacent, which Dolly does not want. Altogether, this gives $6 + 9 + 2 = \boxed{17}$ ways for Dolly to build new houses.

22. We can analyze $2024 \pmod{100}$. Since $4|2024$, only $2024 \pmod{25} \equiv -1 \pmod{25}$ matters. We conclude that the final number will be $1 \pmod{25}$ and $0 \pmod 4$, so the answer must be $\boxed{76}$.

23. Quick trial and error gives that 72 has 12 factors, making it the only number that works. Odd numbers will not work as they have an even amount of factors, as well as the fact that there are no perfect squares with 7 in the tens place. This leaves 72, 74, 76, and 78. None of them are divisible by their amount of factors, except 72. Thus, our answer is $\boxed{72}$.

24. Let h and p be the number of hexagons and pentagons, respectively. We need $6h = 5p$, which means we need to find the least common multiple of 6 and 5. This is 30, so we want 5 hexagons and 6 pentagons to satisfy this equation with the least number of shapes. Our answer is thus $5 + 6 = \boxed{11}$.

25. Because $DE = 3AB$, $EF = 3BC$, and $\angle ABC \equiv \angle DEF$, by SAS similarity, $\triangle ABC \sim \triangle DEF$. Because the side lengths are at a ratio of $1 : 3$ for $\triangle ABC$ to $\triangle DEF$, the areas' ratio must be the square of $1 : 3$, i.e. $1 : 9$. Thus, $\frac{[ABC]}{[DEF]} = \frac{1}{9}$, and since $[DEF] = 18$, the area of $\triangle ABC$ is $\boxed{2}$.

26. From any 5 cards, there are 120 ways to arrange them, and only 1 of these is decreasing. Then, the probability is $\frac{1}{120}$, and the answer is $\boxed{121}$.

27. Rather than computing each value directly, we can recognize that $a_1 = 2^2$ and $a_2 = 2^2$, and that $a_3 = a_2 a_1 = 2^2 \cdot 2^2 = 2^4$. Because of this, we can imagine adding the powers of two that the previous two numbers to get the next number in the sequence. $a_4 = 2^4 \cdot 2^2 = 2^6$, $a_5 = 2^6 \cdot 2^4 = 2^{10}$, $a_6 = 2^{10} \cdot 2^6 = 2^{16}$, $a_7 = 2^{16} \cdot 2^{10} = 2^{26}$, and $a_8 = 2^{26} \cdot 2^{16} = 2^{42}$, making our final answer $\boxed{42}$.

28. We can start by considering the number of ways that Andrea can pick different colored marbles. To do this, Andrea needs 1 red marble, 1 green marble, and 1 blue marble. Since there are 5 red marbles, 3 green marbles, and 2 blue marbles, we can multiply them together to get that there are $5 \cdot 3 \cdot 2 = 30$ ways for different colored marbles to be picked. In total, there are $\binom{10}{3} = 120$ ways of choosing 3 marbles, meaning there is a $\frac{30}{120} = \frac{1}{4}$ chance of her choosing all different colors, making our final answer $1 + 4 = \boxed{5}$.

29. If Cam picks a number higher than 6, he will win automatically, as the highest number Devin can pick is 6. The probability of this is $\frac{4}{10}$. For all other numbers, if Devin and Cam don't tie, the chance Cam wins is $\frac{1}{2}$. The probability that Devin rolls Cam's number is $\frac{1}{6}$, so the probability they don't tie is $\frac{5}{6}$. Then, if Devin and Cam do tie, the probability that Cam wins is the same as the initial probability. So, let S be the probability of Cam winning, then $S = \frac{4}{10} + \frac{6}{10} \cdot \frac{5}{6} \cdot \frac{1}{2} + \frac{6}{10} \cdot \frac{1}{6} \cdot S$. $S = \frac{4}{10} + \frac{1}{4} + \frac{1}{10} \cdot S$. $\frac{9}{10} \cdot S = \frac{13}{20}$. So, $S = \frac{13}{18}$, which gives an answer of $13 + 18 = \boxed{31}$.

30. Statements 2 and 3 cannot be satisfied at the same time, so we consider the other two cases. If the mean and the unique mode both equal 3, $a = b = c = 3$. Otherwise, the median is 4 and the mean is 3, meaning one number has to equal 4 and the other two numbers have to add to 5. The only way to do this while satisfying the median condition is to have the values be 4 and 1. Thus, the maximal possible value of any of the numbers is in this case, with our answer being $\boxed{4}$.

31. Note that at $x = 5$ and above, $x!$ will be a multiple of 10 and thus $(x!)^2$ will be a multiple of 100. Thus, the remainder when the infinite sum is divided by 100 is the same as the remainder when $(1!)^2 + (2!)^2 + (3!)^2 + (4!)^2$ is divided by 100. $(1!)^2 + (2!)^2 + (3!)^2 + (4!)^2 = 1^2 + 2^2 + 6^2 + 24^2 = 1 + 4 + 36 + 576 = 617$, which has a remainder of $\boxed{17}$ when divided by 100.

32. We use casework.

    (a) Case 1: 9 is the longest side. Then the third side must be at least 3, but at most 9, giving 7 options.

    (b) Case 2: The unknown side is the longest side. Then this side must have a length of at least 10 and at most 15, giving 6 options

    So there are a total of $\boxed{13}$ options.

33. Notice that $\triangle MCN \sim \triangle BCD$ with similarity ratio $\frac{1}{2}$. Therefore, we have that the height of $\triangle MCN$ is twice the height of $\triangle BCD$, and hence the height of $\triangle MCN$ is the same as the height of $BMND$. Let this common height be h.

    In addition, notice that $2 \cdot MN = BD$. Therefore, the area of $BMND$ is

$$\frac{MN + BD}{2} \cdot h = \frac{3 \cdot MN \cdot h}{2}.$$

    Now, notice that

$$[ABCD] = [\triangle BCD] - [\triangle BAD] = \frac{BD \cdot 2h}{2} - \frac{BD \cdot h}{2} = 2 \cdot MN \cdot h - MN \cdot h = MN \cdot h.$$

    Therefore, we have that $\frac{2}{3} \cdot [BMND] = [ABCD]$, and hence $[ABCD] = \frac{2}{3} \cdot 21 = \boxed{14}$.

34. If we let $n =$ the number of sides of Tony's shape and $m =$ the number of sides of Daniel's shape, we have at $n = m + 14$, and $(n-2) \cdot 180 = 3 \cdot (m-2) \cdot 180$. We can simplify the second equation to be $n = 3m - 4$, and combining this with the first equation, we get $m + 14 = 3m - 4$, which can be solved for $m = 9$. This means that Daniel's shape has 9 sides, and Tony's shape has $9 + 14 = \boxed{23}$ sides.

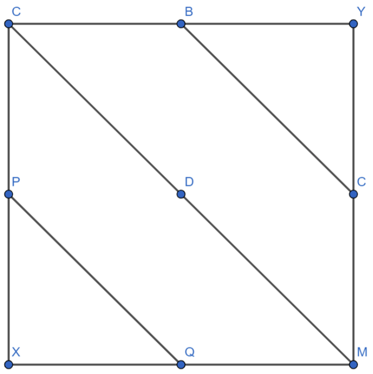
35. The probability can be expressed as $10h^3(1-h)^2$. This is because heads occurs three times, and for the remaining two flips, the probability of tails is $1 - h$. So we square that, but also multiply by $\binom{5}{2} = 10$ as there are 5 flips that occur and the 2 tails can occur whenever. So, we get that $10h^3(1-h)^2 = h^5$. Solving for h, we get that $h = \frac{10 - \sqrt{10}}{9}$, so our answer is $\boxed{29}$.

36. We note that there is 1 digit that is $0 \pmod 3$, 2 digits that are $1 \pmod 3$, and 2 that are $2 \pmod 3$. So, in order for the number to be divisible by 3, either the 3 digits are $0, 1, 2 \pmod 3$ in some order, or the same $\pmod 3$. Thus, in the first case, if the 3 digits are all different mod 3, there are $1 \cdot 2 \cdot 2 = 4$ ways to choose the digits, and there are 3! ways to permute them. This yields a total of 24 numbers here.

    In the second case, the numbers are the same mod 3. If they are all $0 \pmod 3$, there is 1 way to choose the numbers. If they are all 1 or $2 \pmod 3$, there is 2^3 ways to choose the 3 digits. This gives a total of $1 + 8 + 8 = 17$ numbers here.

    So, our answer is $24 + 17 = \boxed{41}$.

37. Drop the altitude from A to D on CM. The area of CAM is $\frac{CM \cdot AD}{2}$. As $CM = 4\sqrt{2}$, we need $AD \geq \sqrt{2}$.



So, draw the lines that are $\sqrt{2}$ away from CM. As the distance from CM to X and Y is $2\sqrt{2}$, the distance from PQ to X is $\sqrt{2}$ and the distance from BC to Y is $\sqrt{2}$. As PQX and BCY are both similar to CMY, we know their sides have a ratio of $\frac{1}{2}$, meaning their areas have a ratio of $\frac{1}{4}$. As CMX and CMY are each $\frac{1}{2}$ the square, the area of PQX and BCY is both $\frac{1}{8}$ of the entirety of the entire square, for a combined $\frac{1}{4} \to \boxed{5}$.

38. For a number to be 10% prime, it would need to have 20 factors (as $\frac{2}{20} = 10\%$). For a number to have 20 factors, its prime factorization must be in one of the forms p_1^{19}, $p_1^9p_2^1$, $p_1^4p_2^3$, or $p_1^4p_2^1p_3^1$ (as $(e_1+1)(e_2+1)(e_3+1)...(e_n+1)$ will have to equal 20, where e_i is the exponent of p_i in its prime factorization). As we want to find the smallest number of this form, we can plug in $2, 3$, and 5 for p_1, p_2, and p_3, giving us values of 2^{19}, 2^93^1, 2^43^3, and $2^43^15^1$. From these values, we can see that $2^43^15^1$ will be the smallest, giving $2^43^15^1 = \boxed{240}$ as our final answer.

39. Notice that the discriminant of the quadratic is

$$(m+n)^2 - 4m(9m+n) = n^2 - 2mn - 35m^2 = (n-7m)(n+5m).$$

Therefore, we wish to find the value of m for which $(n-7m)(n+5m) < 0$ has exactly 551 solutions for n. Finally, notice that all $-5m < n < 7m$ works, and hence $(7m-1)-(-5m+1)+1 = 12m-1$ integer values of n satisfy the inequality. Therefore, we have that $12m-1 = 551$, giving $m = \boxed{46}$.

40. We first notice that $\angle B + \angle D = 180^\circ$. Since the opposite angles of this quadrilateral are supplementary, we know that quadrilateral $ABCD$ is in fact a cyclic quadrilateral. We are also given that $\angle A = \angle C$ so we can easily compute that $\angle A = \angle C = 90^\circ$. Moreover, since $\angle A = 90^\circ$, $\triangle DAB$ is a right triangle. We then use the Pythagorean Theorem to compute that $BD = \sqrt{10^2 + 24^2} = 26$. We know that since quadrilateral $ABCD$ is cyclic, points A, B, C, D are all on the same circle. By using Power of Point, we see that $AE \cdot CE = BE \cdot DE$. Since $BD = 26$ and it's given the midpoint of BD is 5 away from E, The answer is simply $(13+5)(13-5) = \boxed{144}$.