Joe Holbrook Memorial Math Competition

7th Grade

October 20, 2024

1. Observe that $\spadesuit + \spadesuit + \spadesuit = 3 \times \spadesuit = 27$, which implies $\spadesuit = 27 \div 3 = \boxed{9}$.
2. Say he has x friends. Then, $10 - 3x = 1$, so $x = \boxed{3}$.
3. We want the average of 11 and 111, which is $(11 + 111) \div 2 = \boxed{61}$.
4. $(2 + 0 + 2 + 4) \times (20 + 24) = 8 \times 44 = \boxed{352}$.
5. If Brian was asleep for 20% of the test, it means he must have been awake for 80% of the test. Therefore, Brian was awake for $75 \times 0.8 = \boxed{60}$ minutes.
6. $20.24 \times 5 = 101.2$, which, when rounded to the nearest integer, is about $\boxed{101}$.
7. If Gabriele's squares have an area of 4, then the side length of Gabriele's square is 2, as $2 \times 2 = 4$. Then the perimeter of his square is 8. Note that no matter how many squares Gabriele draws, as long as they're all the same, the total perimeter will always be double the total area. Since the total area is 100, the total perimeter will be double that, or $\boxed{200}$.
8. We know 20 feet equals 20×12 inches. So, $\dfrac{20 \times 12}{10}$ is the number of purlunkles you need, or $\boxed{24}$.
9. Because the age difference stays constant, we know that two years ago they were still 16 years apart. Suppose Annabelle's age two years ago was $3x$, and Anastasia's age was x. Then, $3x - x = 2x = 16$, which means $x = 8$. Thus, Annabelle was 24 and Anastasia was 8 two years ago. Now, they are 26 and 10. Let y be the number of years until Annabelle is 2 times Anastasia's age. Then, $26 + y = 2(10 + y)$. So, $y = \boxed{6}$ years.
10. Let the side length of one square be x. Then, the square will have diagonal length $x\sqrt{2}$, and since $x\sqrt{2} > x$, the smaller square will have side length x and the larger square with have side length $x\sqrt{2}$. Hence, the desired answer is
$$\frac{(x\sqrt{2})^2}{x^2} = \frac{2x^2}{x^2} = \boxed{2}.$$
11. Every visit, the day of the week advances by 3 days, and we need to move 1 day ahead of Saturday. This occurs after 5 visits, because $5 \cdot 3 = 15$ leaves a remainder of 1 when divided by 7. Thus, after $\boxed{85}$ days, the day will be Sunday.
12. The only two digit palindromes that becomes a three digit number when 13 is added to it are 88 or 99, and testing shows that only $x = \boxed{88}$ works.
13. Using the Pythagorean theorem, Amy must walk $\sqrt{12^2 + 16^2} = 20$ feet. Sabrina walks a total of $12+16 = 28$ feet, so the difference is $28 - 20 = \boxed{8}$.
14. Consider the worst case, in which he picks distinct pairs of socks right until his last pick. This would mean that for 7 picks in a row he needs distinct socks, and therefore $\boxed{7}$ pairs of distinct socks. Notice that on his 8th pick it does not matter what sock he picks, as whatever he picks it will be a pair with another one already chosen. This situation can also be modeled with 7 pigeonholes and 8 pigeons.
15. Let h and p be the number of hexagons and pentagons, respectively. We need $6h = 5p$, which means we need to find the least common multiple of 6 and 5. This is 30, so we want 5 hexagons and 6 pentagons to satisfy this equation with the least number of shapes. Our answer is thus $5 + 6 = \boxed{11}$.
16. Increasing the length multiplies the area by $\frac{5}{4}$, so we must multiply the width by $\frac{8}{5}$ to double the area, since $\frac{5}{4} \cdot \frac{8}{5} = 2$. $\frac{8}{5}$ equals 160 percent, so the answer is $\boxed{60}$.

17. Using the difference of squares factorization,

$$41^2 - 4 = (41 - 2)(41 + 2) = 39 \cdot 43.$$

39 has remainder 4 when divided by 7, and 43 has remainder 1, so their product has remainder $\boxed{4}$.

18. Since there are exactly 3 colors visible, $2 < \frac{3}{10}n \leq 3$. Solving each of these inequalities separately, $\frac{10}{3} \cdot 2 < n$ and $n \leq \frac{10}{3} \cdot 3$, or $\frac{20}{3} < n \leq 10$. Since n must be an integer, $n = 7, 8, 9$, or 10. The sum of these is $\boxed{34}$.

19. It takes Jeremy 15 minutes to walk half the way to Andrea's house. Therefore, it took Jeremy 5 minutes to bike the same distance. So, Jeremy bikes $15 \div 5 = 3$ times faster than he walks. So, his biking speed is $\boxed{6}$ miles an hour.

20. We factor out 2^{2022} to get $(2^{2022})(4 - 1)$. 2^{2022} only has 2 as a prime divisor, while $4 - 1 = 3$ has 3 as a prime divisor. So, the largest prime factor of the expression is $\boxed{3}$.

21. Because this number is in base 8, this means that all digits are less than or equal to 7. Since the product of the digits is 40, the factorization is $2^3 \cdot 5$. Thus, as we want the least number of digits in the number as well, the smallest base 8 number this can be is 245_8. In base 10, this is $2 \cdot 8^2 + 4 \cdot 8 + 5 = \boxed{165}$.

22. If some person takes the red marble, there are only 3 people who therefore cannot have taken the blue marble: the person who took the red marble and the two people sitting next to them. Therefore, for each of the 26 people who could take the red marble, 23 others could have taken the blue marble. Every other person will have a green marble. Thus, there are $26 * 23 = \boxed{598}$ distinct ways to distribute the marbles.

23. If we let $n =$ the number of sides of Tony's shape and $m =$ the number of sides of Daniel's shape, we have at $n = m + 14$, and $(n - 2) \times 180 = 3 \times (m - 2) \times 180$. We can simplify the second equation to be $n = 3m - 4$, and combining this with the first equation, we get $m + 14 = 3m - 4$, which can be solved for $m = 9$. This means that Daniel's shape has 9 sides, and Tony's shape has $9 + 14 = \boxed{23}$ sides.

24. From any 5 cards, there are 120 ways to arrange them, and only 1 of these is decreasing. Then, the probability is $\frac{1}{120}$, and the answer is $\boxed{121}$.

25. Statements 2 and 3 cannot be satisfied at the same time, so we consider the other two cases. If the mean and the unique mode both equal 3, $a = b = c = 3$. Otherwise, the median is 4 and the mean is 3, meaning one number has to equal 4 and the other two numbers have to add to 5. The only way to do this while satisfying the median condition is to have the values be 4 and 1. Thus, the maximal possible value of any of the numbers is in this case, with our answer being $\boxed{4}$.

26. The volume of a cone is $\frac{r^2 h\pi}{3}$, and plugging in $h = 100$ we get that the volume of the cone is $\frac{100r^2\pi}{3}$, and the volume of the sphere will be $\frac{4}{3}\pi r^3$. If we set these equal to each other, we get $\frac{100r^2\pi}{3} = \frac{4}{3}\pi r^3$ which can be solved to get $r = \boxed{25}$.

27. We can start by considering the number of ways that Andrea can pick different colored marbles. To do this, Andrea needs 1 red marble, 1 green marble, and 1 blue marble. Since there are 5 red marbles, 3 green marbles, and 2 blue marbles, we can multiply them together to get that there are $5 \times 3 \times 2 = 30$ ways for different colored marbles to be picked. In total, there are $\binom{10}{3} = 120$ ways of choosing 3 marbles, meaning there is a $\frac{30}{120} = \frac{1}{4}$ chance of her choosing all different colors, making our final answer $1+4 = \boxed{5}$.

28. We can first see that B, D, and F must all be even and D must be either 0 or 5. Since we are trying to find the minimum number, we can first assume that $A = 1$ to try to find a minimal case. One thing that we also need to realize is that $3|A + B + C$ and $3|A + B + C + D + E + F$. Since we know $3|A + B + C$, $3|D + E + F$. In order to find a small value of $\overline{ABCDEF}$, we need to try to minimize the values in the higher digits. Since there is no other restriction for B except it must be even, we can also set $B = 0$. As $A = 1$ and $B = 0$, $C = 2, 5, 8$. Since C is the next biggest number we can control, we set C as 2 to minimize it. The last condition we have to keep is that $4|\overline{CD}$ so $D = 0, 4, 8$. Once we set $D = 0$, we can see that we can automatically set $E = F = 0$ and see that this satisfies all the conditions. Our answer will be $\boxed{102000}$

29. The range of sums that Jacob could have is from 2 to 12 so the only perfect squares he can roll as a sum are 4 and 9. For 4, we can roll it in three ways, which gives a probability of $\frac{3}{36}$. For 9, we can roll it in four ways, which gives a probability of $\frac{4}{36}$. Adding these together, we have a total probability of $\frac{7}{36}$. So, $p = 7$, $q = 36$, and $p + q = \boxed{43}$.

30. At the end of one hour, the expected number of oranges that were added would be $1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{3} = \frac{7}{6}$. By the linearity of expectation, at the end of 10 hours, he will have added an expected number of $10 \cdot \frac{7}{6} = \frac{35}{3}$. Added to his original 3 oranges, this gives a total of $\frac{44}{3}$ oranges. Thus, the answer is $\boxed{47}$.

31. Note that at $x = 5$ and above, $x!$ will be a multiple of 10 and thus $(x!)^2$ will be a multiple of 100. Thus, the remainder when the infinite sum is divided by 100 is the same as the remainder when $(1!)^2 + (2!)^2 + (3!)^2 + (4!)^2$ is divided by 100. $(1!)^2 + (2!)^2 + (3!)^2 + (4!)^2 = 1^2 + 2^2 + 6^2 + 24^2 = 1 + 4 + 36 + 576 = 617$, which has a remainder of $\boxed{17}$ when divided by 100.

32. Notice that there are 5 even numbers in total we have to place, so they have to go on the corners of the grid and the center. This means we have to still place 2, 6, and 8. Because 6 and 9 are not relatively prime, 6 has to go in the corner cells in the 3rd column. We can assume 2 and 8 are interchangeable since they are powers of 2, so we have 2 choices to place them. 3 is predetermined because there is only one unfilled cell not adjacent to 6. This leaves 5 and 7 to place. They are interchangeable in one case when 10 has all of its neighbors, and fixed in the other. So, our answer, adding up the 2 cases when looking at where 6 can go, is $2 + 2 \cdot 2 = \boxed{6}$.

33. Let O be the center of the circumcircle of $ABCDE$.

    Since AD is a diameter of the circle, we have that $\triangle ADE$ is a right triangle with hypotenuse AD by Thales Theorem. Therefore, we have that since $DE = 6$ and $AE = 8$, we have that $AD = 10$. Hence, we have that $AO = BO = CO = 5$.

    Now, notice that $\triangle AOB$ and $\triangle COD$ are equilateral, and hence $\angle AOB = \angle COD = 60^\circ$. Therefore, we have that $\angle BOC = 60^\circ$. Finally, we also have that $BO = CO = 5$, and hence $\triangle BOC$ is isosceles. Therefore, we have that $\angle CBO = \angle BCO = 60^\circ$, meaning that $\triangle BOC$ is also equilateral. Hence, we have that $BC = \boxed{5}$.

34. The probability can be expressed as $10h^3(1-h)^2$. This is because heads occurs three times, and for the remaining two flips, the probability of tails is $1 - h$. So we square that, but also multiply by $\binom{5}{2} = 10$ as there are 5 flips that occur and the 2 tails can occur whenever. So, we get that $10h^3(1-h)^2 = h^5$. Solving for h, we get that $h = \frac{10 - \sqrt{10}}{9}$, so our answer is $\boxed{29}$.

35. We note that there is 1 digit that is $0 \pmod 3$, 2 digits that are $1 \pmod 3$, and 2 that are $2 \pmod 3$. So, in order for the number to be divisible by 3, either the 3 digits are $0, 1, 2 \pmod 3$ in some order, or the same $\pmod 3$. Thus, in the first case, if the 3 digits are all different mod 3, there are $1 \times 2 \times 2 = 4$ ways to choose the digits, and there are 3! ways to permute them. This yields a total of 24 numbers here.

    In the second case, the numbers are the same mod 3. If they are all $0 \pmod 3$, there is 1 way to choose the numbers. If they are all 1 or $2 \pmod 3$, there is 2^3 ways to choose the 3 digits. This gives a total of $1 + 8 + 8 = 17$ numbers here.

    So, our answer is $24 + 17 = \boxed{41}$.

36. One case where this works is when either the max or the minimum number is divisible by 4. In this range, only 4 itself would work. If 4 is the maximum, there are 2^3 subsets, and if 4 is the minimum, then there are 2^3 subsets as well. But, we overcount the case where 4 is the only number, so the total subsets here is $2^3 + 2^3 - 1 = 15$. The only case not accounted for is if the minimum and maximum are either 2 or 6. If the minimum is 2 and the maximum is 6, we get 2^3 subsets. If the minimum and maximum are both 2, or the minimum and maximum are both 6, then we get the sets with one element $\{2\}$ and $\{6\}$, giving us 2 extra sets. Summing, we have $15 + 8 + 2 = \boxed{25}$ subsets.

37. For a number to be 10% prime, it would need to have 20 factors (as $\frac{2}{20} = 10\%$). For a number to have 20 factors, its prime factorization must be in one of the forms p_1^{19}, $p_1^9 p_2^1$, $p_1^4 p_2^3$, or $p_1^4 p_2^1 p_3^1$ (as $(e_1 + 1)(e_2 +$

$1)(e_3+1)...(e_n+1)$ will have to equal 20, where e_i is the exponent of p_i in its prime factorization). As we want to find the smallest number of this form, we can plug in $2, 3$, and 5 for p_1, p_2, and p_3, giving us values of 2^{19}, $2^9 3^1$, $2^4 3^3$, and $2^4 3^1 5^1$. From these values, we can see that $2^4 3^1 5^1$ will be the smallest, giving $2^4 3^1 5^1 = \boxed{240}$ as our final answer.

38. Parity argument gives that one of the primes, WLOG r, is 2, since 617 is odd and the sum of four odd numbers is even. Therefore, $pq + 2q + 2p + 2pq = 617$, so $3pq + 2q + 2p = 617$, so $9pq + 6q + 6p = 1851$, and $9pq + 6q + 6p + 4 = 1855$. Thus $(3p + 2)(3q + 2) = 1855 = 5 \cdot 7 \cdot 53$, so the only possibility is p and q are 17 and 11. Our answer is $2 + 11 + 17 = \boxed{30}$.

39. We first notice that $\angle B + \angle D = 180^\circ$ so the sum of angles A and C must be $360 - 180 = 180^\circ$. We are also given that $\angle A = \angle C$ so we can easily compute that $\angle A = \angle C = 90^\circ$. Since the opposite angles of this quadrilateral are supplementary, we know that quadrilateral $ABCD$ is in fact a cyclic quadrilateral. Moreover, since $\angle A = 90^\circ$, $\triangle DAB$ is a right triangle. We then use the Pythagorean Theorem to compute that $BD = \sqrt{10^2 + 24^2} = 26$. We know that since quadrilateral $ABCD$ is cyclic, points A, B, C, D are all on the same circle. By using Power of Point, we see that $AE \times CE = BE \times DE$. We have to notice that $\triangle DAB$ and $\triangle DCB$ both have the same circumcircle as they share $\overline{BD}$ as the hypotenuse and are right triangles (One can also notice this as quadrilateral $ABCD$ is cyclic so all the points lie on the same circle). The diameter of the circumcircle is $\overline{BD}$ and the midpoint of $\overline{BD}$, point O, is the center of the circle. Since $BD = 26$, the length of the radius r is half of the diameter so $r = 13$. Now, $BE = r + OE$ and $DE = r - OE$ so $BE \times DE = (r + OE) \times (r - OE) = r^2 - OE^2 = 13^2 - 5^2 = 169 - 25 = \boxed{144}$.

40. All integers less than 3^7 have at most 7 digits in base 3. Furthermore, the problem becomes how can we arrange 0s, 1s, and 2s into 7 positions with two consecutive 1s.

    However, this is much easier to count if we instead find the number of such arrangements that do not have two consecutive 1s. Let $F(n)$ be the number of such arrangements in n positions. For an arrangement of length $n + 1$ use casework on the first position:

    (a) Case 1: The first digit is 0. Then the next digit can be anything, so we just need to find the number of arrangements of length n, which is $F(n)$.

    (b) Case 2: The first digit is 1. Then the next digit must be a 0 or 2, after which we can have anything. So, the number of arrangements is $2 \cdot F(n - 1)$.

    (c) Case 3: The first digit is 2. This is the same as the first case, so there are $F(n)$ such arrangements.

    Summing this up, we get that $F(n + 1) = 2 \cdot F(n) + 2 \cdot F(n - 1)$. As $F(1) = 3, F(2) = 8$, we get that $F(3) = 22, F(4) = 60, F(5) = 164, F(6) = 448, F(7) = 1224$. So, the number of arrangements of length 7 without two consecutive 1s is 1224. As there are 2187 total arrangements, there are $2187 - 1224 = \boxed{963}$ arrangements with two consecutive 1s.