Joe Holbrook Memorial Math Competition

8th Grade

October 20, 2024

1. If Jeremy randomly selects all of the consonants before any vowels, he will pick 20 letters. His next letter, the 21st letter, must necessarily be a vowel. Because this is the worst possible scenario, Jeremy must select $\boxed{21}$ random letters to ensure he picks at least one vowel.

2. $(2+0+2+4)\times(20+24) = 8\times 44 = \boxed{352}$.

3. Consider a right isosceles triangle with a base the same length as the distance between NYC and Zero Ground. This implies that the side lengths will be $7\sqrt{2}$, and using the Pythagoras theorem, we get that the base is $\sqrt{49\cdot 2+49\cdot 2} = \boxed{14}$

4. Arnav has a $\frac{1}{10}$ probability of hitting a bullseye on each shot. Since the shots are independent, the probabilities multiply. The probability of getting two bullseyes in a row is $\frac{1}{10}\times\frac{1}{10} = \frac{1}{100}$, or $\boxed{101}$ as desired.

5. Using the definition of the function, $6\#((9\#3)\#4) = 6\#((\frac{9+3}{3})\#4) = 6\#(4\#4) = 6\#(\frac{4+4}{4}) = 6\#2 = \frac{6+2}{2} = \boxed{4}$.

6. We can rearrange the expression to $26^2-25^2+31^2-30^2$ which can be expressed as $(26+25)(26-25)+(31+30)(31-30) = 51+61 = \boxed{112}$

7. Every visit, the day of the week advances by 3 days, and we need to move 1 day ahead of Saturday. This occurs after 5 visits, because $5\cdot 3 = 15$ leaves a remainder of 1 when divided by 7. Thus, after $\boxed{85}$ days, the day will be Sunday.

8. We write an equation with x as the number of hours he spends driving at 90 miles per hour: $\frac{80+90x}{1+x} = 84$. This gets $x = \frac{2}{3}$ hours, or 40 minutes, meaning in total he drove for $40+60 = \boxed{100}$ minutes .

9. Out of the original 20 games, Cristiano has won $20\times 0.4 = 8$ of the games. We can write the fraction $\frac{8}{20}$, with the numerator being the amount of games Cristiano has won and the denominator being the total amount of games Cristiano has played. Let x be the amount of extra games Cristiano plays. Since Cristiano needs to win at least 75% of his total games, we can write the expression $\frac{8+x}{20+x} = 0.75 = \frac{0.75}{1}$. We can cross multiply to get $8+x = 15+0.75x$. Simplify to get $0.25x = 7$, and then $x = \boxed{28}$.

10. We factor out 2^{2022} to get $(2^{2022})(4-1)$. 2^{2022} only has 2 as a prime divisor, while $4-1=3$ has 3 as a prime divisor. So, the largest prime factor of the expression is $\boxed{3}$.

11. Using the Pythagorean theorem, Amy must walk $\sqrt{12^2+16^2} = 20$ feet. Sabrina walks a total of $12+16 = 28$ feet, so the difference is $28-20 = \boxed{8}$.

12. Since we want to keep the physics textbooks together, we can treat them simply as one book. There are $\frac{8!}{4!3!} = \boxed{280}$ ways to line his books up.

13. Quick trial and error gives that 72 has 12 factors, making it the only number that works. Odd numbers will not work as they have an even amount of factors, as well as the fact that there are no perfect squares with 7 in the tens place. This leaves 72, 74, 76, and 78. None of them are divisible by their amount of factors, except 72. Thus, our answer is $\boxed{72}$.

14. Let h and p be the number of hexagons and pentagons, respectively. We need $6h = 5p$, which means we need to find the least common multiple of 6 and 5. This is 30, so we want 5 hexagons and 6 pentagons to satisfy this equation with the least number of shapes. Our answer is thus $5 + 6 = \boxed{11}$.

15. Increasing the length multiplies the area by $\frac{5}{4}$, so we must multiply the width by $\frac{8}{5}$ to double the area, since $\frac{5}{4} \cdot \frac{8}{5} = 2$. $\frac{8}{5}$ equals 160 percent, so the answer is $\boxed{60}$.

16. Because this number is in base 8, this means that all digits are less than or equal to 7. Since the product of the digits is 40, the factorization is $2^3 \cdot 5$. Thus, as we want the least number of digits in the number as well, the smallest base 8 number this can be is 245_8. In base 10, this is $2 \cdot 8^2 + 4 \cdot 8 + 5 = \boxed{165}$.

17. We can analyze $2024 \pmod{100}$. Since $4|2024$, only $2024 \pmod{25} \equiv -1 \pmod{25}$ matters. We conclude that the final number will be $1 \pmod{25}$ and $0 \pmod{4}$, so the answer must be $\boxed{76}$.

18. Since the volume of the balloon multiplied by 8, the surface area multiplied by $8^{\frac{2}{3}}$, or 4 times, and thus a new surface area of $\boxed{128}$ square inches.

19. We can start by considering the number of ways that Andrea can pick different colored marbles. To do this, Andrea needs 1 red marble, 1 green marble, and 1 blue marble. Since there are 5 red marbles, 3 green marbles, and 2 blue marbles, we can multiply them together to get that there are $5 \times 3 \times 2 = 30$ ways for different colored marbles to be picked. In total, there are $\binom{10}{3} = 120$ ways of choosing 3 marbles, meaning there is a $\dfrac{30}{120} = \dfrac{1}{4}$ chance of her choosing all different colors, making our final answer $1+4 = \boxed{5}$.

20. If we let $n =$ the number of sides of Tony's shape and $m =$ the number of sides of Daniel's shape, we have at $n = m + 14$, and $(n - 2) \times 180 = 3 \times (m - 2) \times 180$. We can simplify the second equation to be $n = 3m - 4$, and combining this with the first equation, we get $m + 14 = 3m - 4$, which can be solved for $m = 9$. This means that Daniel's shape has 9 sides, and Tony's shape has $9 + 14 = \boxed{23}$ sides.

21. The difference in speed between Magikarp and Bulbasaur is $80 - 40 = 40$ meters per hour. Thus, every hour they are distanced by 40 meters. In order for the two to meet each other, Bulbasaur needs to outrun Magikarp by one lap which is 400 meters. For this to happen 69 times, Bulbasaur needs to outrun Magikarp by $400 \times 69 = 27600$ meters. Since Bulbasaur outruns Magikarp by 40 meters every hour, it will take them $\dfrac{27600}{40} = \boxed{690}$ hours

22. Let r, b, and g be the number of red, blue, and green crayons, respectively. Then, we can set up the following equations:

    (a) $b + g = 7$

    (b) $r + g = 3$

    (c) $r + b = 8$

    Adding up all the equations, we get $2r + 2b + 2g = 18$. Therefore, the total number of crayons, $r + b + g$, equals $\boxed{9}$.

23. Because $DE = 3AB$, $EF = 3BC$, and $\angle ABC \equiv \angle DEF$, by SAS similarity, $\triangle ABC \sim \triangle DEF$. Because the side lengths are at a ratio of $1 : 3$ for $\triangle ABC$ to $\triangle DEF$, the areas' ratio must be the square of $1 : 3$, i.e. $1 : 9$. Thus, $\dfrac{[ABC]}{[DEF]} = \dfrac{1}{9}$, and since $[DEF] = 18$, the area of $\triangle ABC$ is $\boxed{2}$.

24. The numbers in base 5 with 5 digits range from 10000_5 to 44444_5, which become 625 and 3124 when converted to base 10. The numbers in base 7 with 5 digits similarly range from 10000_7 to 66666_7, which are 2401 and 16806 in base 10. The overlap between these two intervals is from 2401 to 3124. So, the sum is $2401 + 3124 = \boxed{5525}$.

25. Using Simon's Favorite Factoring Trick:

    $$6ab - 3a + 2b = (3a + 1) \cdot (2b - 1) + 1 = 29$$

    $$(3a + 1) \cdot (2b - 1) = 28$$

    As a, b are positive integers, We notice that $3a + 1$ and $2b - 1$ must be positive factors of 28. As $2b - 1$ is always odd, $2b - 1 = 1, 7$. If $2b - 1 = 1$, then $3a + 1 = 28$, giving $(a, b) = (9, 1)$. If $2b - 1 = 7$, then $3a + 1 = 4$, giving $(a, b) = (1, 4)$. So, the possible ab are 9 and 4, for a sum of $\boxed{13}$.

26. Notice that in order to reach Farmer John in 12 steps, Bessie will need to first reach one of $(6, 3)$ or $(3, 6)$, after which she will have only one way to reach Farmer John with her remaining 3 moves. As a result, we can now think of this problem as the number of ways to reach one of these two points.

    Considering $(6, 3)$, the only way Bessie can reach this point within 9 moves is if she moves right 6 times and up 3 times in some order. If we think about this being a list of 9 moves and for us to choose 6 of them to be right moves and the remaining 3 be up moves, we can say that there are $\binom{9}{6} = 84$ ways of doing this. We can do a similar calculation to get that there are 84 ways of reaching $(3, 6)$, meaning in total, Bessie will have $84 + 84 = \boxed{168}$ ways of reaching Farmer John.

27. Note that at $x = 5$ and above, $x!$ will be a multiple of 10 and thus $(x!)^2$ will be a multiple of 100. Thus, the remainder when the infinite sum is divided by 100 is the same as the remainder when $(1!)^2 + (2!)^2 + (3!)^2 + (4!)^2$ is divided by 100. $(1!)^2 + (2!)^2 + (3!)^2 + (4!)^2 = 1^2 + 2^2 + 6^2 + 24^2 = 1 + 4 + 36 + 576 = 617$, which has a remainder of $\boxed{17}$ when divided by 100.

28. The range of sums that Jacob could have is from 2 to 12 so the only perfect squares he can roll as a sum are 4 and 9. For 4, we can roll it in three ways, which gives a probability of $\frac{3}{36}$. For 9, we can roll it in four ways, which gives a probability of $\frac{4}{36}$. Adding these together, we have a total probability of $\frac{7}{36}$. So, $p = 7$, $q = 36$, and $p + q = \boxed{43}$.

29. Notice that there are 5 even numbers in total we have to place, so they have to go on the corners of the grid and the center. This means we have to still place 2, 6, and 8. Because 6 and 9 are not relatively prime, 6 has to go in the corner cells in the 3rd column. We can assume 2 and 8 are interchangeable since they are powers of 2, so we have 2 choices to place them. 3 is predetermined because there is only one unfilled cell not adjacent to 6. This leaves 5 and 7 to place. They are interchangeable in one case when 10 has all of its neighbors, and fixed in the other. So, our answer, adding up the 2 cases when looking at where 6 can go, is $2 + 2 \cdot 2 = \boxed{6}$.

30. Notice that
$$\angle QPR = \angle QPO + \angle RPO = \frac{90^\circ}{2} + \frac{90^\circ}{2} = 90^\circ.$$
    Therefore, by the Inscribed Angle Theorem, we have that $QR = 2 \cdot 90^\circ = 180^\circ$. Therefore, QR is a diameter of the circle, and hence the radius of the circle is half of its length, which is $\frac{2024}{2} = \boxed{1012}$.

31. The probability can be expressed as $10h^3(1-h)^2$. This is because heads occurs three times, and for the remaining two flips, the probability of tails is $1 - h$. So we square that, but also multiply by $\binom{5}{2} = 10$ as there are 5 flips that occur and the 2 tails can occur whenever. So, we get that $10h^3(1-h)^2 = h^5$. Solving for h, we get that $h = \frac{10 - \sqrt{10}}{9}$, so our answer is $\boxed{29}$.

32. We note that there is 1 digit that is $0 \pmod 3$, 2 digits that are $1 \pmod 3$, and 2 that are $2 \pmod 3$. So, in order for the number to be divisible by 3, either the 3 digits are $0, 1, 2 \pmod 3$ in some order, or the same $\pmod 3$. Thus, in the first case, if the 3 digits are all different mod 3, there are $1 \times 2 \times 2 = 4$ ways to choose the digits, and there are 3! ways to permute them. This yields a total of 24 numbers here.

    In the second case, the numbers are the same mod 3. If they are all $0 \pmod 3$, there is 1 way to choose the numbers. If they are all 1 or $2 \pmod 3$, there is 2^3 ways to choose the 3 digits. This gives a total of $1 + 8 + 8 = 17$ numbers here.

    So, our answer is $24 + 17 = \boxed{41}$.

33. For a number to be 10% prime, it would need to have 20 factors (as $\frac{2}{20} = 10\%$). For a number to have 20 factors, its prime factorization must be in one of the forms p_1^{19}, $p_1^9 p_2^1$, $p_1^4 p_2^3$, or $p_1^4 p_2^1 p_3^1$ (as $(e_1 + 1)(e_2 + 1)(e_3 + 1)...(e_n + 1)$ will have to equal 20, where e_i is the exponent of p_i in its prime factorization). As we want to find the smallest number of this form, we can plug in $2, 3$, and 5 for p_1, p_2, and p_3, giving us values of 2^{19}, $2^9 3^1$, $2^4 3^3$, and $2^4 3^1 5^1$. From these values, we can see that $2^4 3^1 5^1$ will be the smallest, giving $2^4 3^1 5^1 = \boxed{240}$ as our final answer.

34. As AC is the diameter, we know ABC is a right triangle so

$$AC = \sqrt{AB^2 + BC^2} = \sqrt{8^2 + 15^2} = 17.$$

Next, note that as the ray from B bisects the arc AC, it is an angle bisector of $\angle ABC$. Thus, we can use the Angle-Bisector Theorem to finish this problem. Let $AD = x$. Then

$$\frac{AB}{BC} = \frac{AD}{DC}$$

or

$$\frac{8}{15} = \frac{x}{17 - x}.$$

Cross-multiplying gives $8(17 - x) = 15x$, which we can solve to get $23x = 136$ and $x = \frac{136}{23}$. So, $m = 136$ and $n = 23$ and $m + n = \boxed{159}$.

35. One case where this works is when either the max or the minimum number is divisible by 4. In this range, only 4 itself would work. If 4 is the maximum, there are 2^3 subsets, and if 4 is the minimum, then there are 2^3 subsets as well. But, we overcount the case where 4 is the only number, so the total subsets here is $2^3 + 2^3 - 1 = 15$. The only case not accounted for is if the minimum and maximum are either 2 or 6. If the minimum is 2 and the maximum is 6, we get 2^3 subsets. If the minimum and maximum are both 2, or the minimum and maximum are both 6, then we get the sets with one element $\{2\}$ and $\{6\}$, giving us 2 extra sets. Summing, we have $15 + 8 + 2 = \boxed{25}$ subsets.

36. Parity argument gives that one of the primes, WLOG r, is 2, since 617 is odd and the sum of four odd numbers is even. Therefore, $pq + 2q + 2p + 2pq = 617$, so $3pq + 2q + 2p = 617$, so $9pq + 6q + 6p = 1851$, and $9pq + 6q + 6p + 4 = 1855$. Thus $(3p + 2)(3q + 2) = 1855 = 5 \cdot 7 \cdot 53$, so the only possibility is p and q are 17 and 11. Our answer is $2 + 11 + 17 = \boxed{30}$.

37. While at first it may seem that the answer depends on the height BC of the rectangle, we will see that it does not. Let $AB = w$ and $BC = h$. Then, in order to satisfy the relation, we must have $AP = \frac{5}{11}w$ and $PB = \frac{6}{11}w$. We can now find all of the areas. We have

$$[APD] = \frac{1}{2} \cdot h \cdot \frac{5}{11}w = \frac{5}{22}hw,$$
$$[DPC] = \frac{1}{2} \cdot h \cdot w = \frac{1}{2}hw,$$
$$[CPB] = \frac{1}{2} \cdot h \cdot \frac{6}{11}w = \frac{6}{22}hw.$$

The ratio we are looking for is $\frac{[DPC]}{[APD]} = \frac{1/2}{5/22} = \frac{11}{5}$, so the answer is $11 + 5 = \boxed{16}$.

38. Let x be the price of the first item expressed as a decimal in terms of dollars, and let y be the price of the second item expressed as a decimal in terms of dollars.

Notice that $xy = x + y$, and rearranging and using SFFT gives $(x - 1)(y - 1) = 1$, meaning that $x - 1 = a$ and $y - 1 = \frac{1}{a}$ for some real number a. Then, since $x = 1 + a$ and x is a positive integer, we have that a is also a nonnegative integer. In addition, since $y = 1 + \frac{1}{a}$ and y is a decimal with at most 2 decimal places after the decimal point, we have that a must be a factor of 100.

Finally, $100 = 2^2 \cdot 3^2$ has 9 factors, but $a = 1$ doesn't work as it would cause y to also be an integer, while the rest of the possible values of a give valid prices. Therefore, there are $9 - 1 = \boxed{8}$ possible prices for the two products.

39. Triangle ABC must be a right triangle with angle A being right. To see why, note that we are given

$$[ABC] = AD \cdot AE = \frac{1}{2}BC \cdot AD \implies AE = \frac{1}{2}BC = BE = EC.$$

Next, since $AE = BE = CE$, we know E is the center of the circle passing through points A, B, C. In this circle, BC is a diameter (as it consists of two radii), so $\angle BAC = 90°$.

Thus, we have $BC = \sqrt{5^2 + 6^2} = \sqrt{61}$. Then,

$$[ABC] = \frac{1}{2}AB \cdot AC = \frac{30}{2} = \frac{1}{2}BC \cdot AD \implies AD = \frac{30}{\sqrt{61}}.$$

Squaring this, we get $AD^2 = \frac{30^2}{61}$, so the answer is $30^2 + 61 = 900 + 61 = \boxed{961}$.

40. We know this is equivalent to:

$$\sum_{cyc} \sum_{a=0}^{\infty} \frac{1}{3^a} \sum_{b=0}^{\infty} \frac{1}{3^b} \sum_{c=0}^{\infty} \frac{c}{3^c} \tag{1}$$

The first term indicates 3 copies of the following sum. The middle 2 sums each have value $\frac{1}{1 - \frac{1}{3}} = \frac{3}{2}$. Let the last sum have value S, then $S - \frac{S}{3} = \frac{\frac{1}{3}}{1 - \frac{1}{3}} = \frac{1}{2}$. Then, we get $S = \frac{3}{4}$. Finally, the product is $3 \cdot (\frac{3}{2})^2 \cdot (\frac{3}{4}) = \frac{3^4}{2^4}$. The desired answer is $3^4 + 2^4 = \boxed{97}$.